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DYNAMIC PROGRAMMING

The future ainâĂŹt what it used to be.

Yogi Berra

Contents

Mathematical Preliminaries

In this first introductory chapter, we introduce some mathematical results that will be useful later on in the course. We'll see several intermediate value theorems, some calculus results, Leibniz' rules for differentiation under the integral sign, properties of concave and convex functions, static optimization results and the envelope theorem.

Intermediate value theorems

The intermediate value theorem is a basic result in mathematical analysis. It states that if a continuous real valued function (on a closed interval) reaches two values, it also attains every intermediate value.

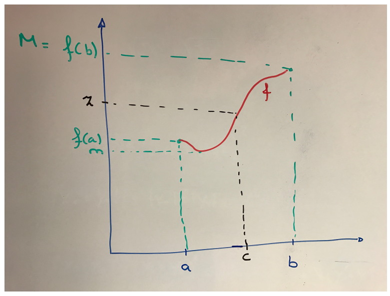

Figure 1: Illustration of the intermediate value theorem.
Notice that c is not necessarily unique: there might be more than one value in $[a, b]$ that attains the value z.

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the interval $[a, b]$ and assume that the range $f([a, b])$ includes the values m and M with $m < M$, then for all z with $m \leq z \leq M$, there is a $c \in [a, b]$ such that $f(c) = z$.*

Proof. Let $x_m, x_M \in [a, b]$ such that $f(x_m) = m$ and $f(x_M) = M$.

Case 1: $x_m < x_M$

Consider the set $S = \{x \in [x_m, x_M] | f(x) < z\}$. The set is bounded from above by x_M so S has a supremum (least upper bound), say c.

By the properties of a supremum, for all t, there is a $x_t \in S$ such that $c - 1/t < x_t \leq c$, otherwise c is not the lowest upperbound. In addition there is an $y_t \notin S, y_t \leq x_M$ such that $c \leq y_t \leq c + 1/t$, otherwise c is not an upperbound. Observe that $f(x_t) < z$ and $z \leq f(y_t)$, so.

$$f(x_t) < z \leq f(y_t),$$

As both $x_t \to c$ and $y_t \to c$ it follows that, $f(c) \leq z \leq f(c)$, so $f(c) = z$.

Case 2: $x_m > x_M$

This is very similar to the first part of the proof. Try it for yourself.

□

THE FIRST VARIATION on the intermediate value theorem is the **integrated mean value theorem**. It states the for a continuous function f on a closed interval. the area underneath f is always equal to the size of a rectangle defined by the base $[a, b]$ and the height of $f(c)$ for some c between a and b.

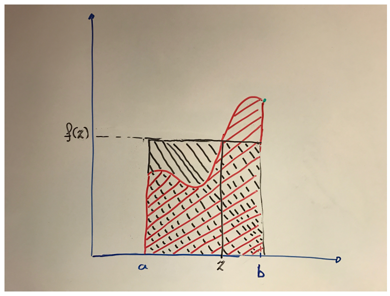

Figure 2: Illustration of the integrated mean value theorem.
Again, notice that z is not necessarily unique.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the interval $[a, b]$ then there is a $z \in [a, b]$ such that $\int_a^b f(t)dt = f(z)(b-a)$.*

Proof. The function $f(x)$ is continuous on the compact set $[a, b]$ as such, it reaches a maximum and minimum on this set, say m and M. Then, for all $x \in [a, b]$, $m \leq f(x) \leq M$, so

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a),$$

So,

$$m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M,$$

From the intermediate value theorem there is a $z \in [a, b]$ such that $f(z) = \frac{1}{b-a} \int_a^b f(x)dx$ as was to be shown. □

Above theorem also shows that if f is C^1, then there is a $z \in [a, b]$ such that

$$f(b) - f(a) = f'(z)(b-a).$$

AS A FINAL variation on the intermediate value theorem, we present the weighted mean value theorem.[1]

[1] This theorem is also used to show the validity of Taylor expansions.

Theorem 3. *If $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and g does not change sign on $[a, b]$ then there is a $z \in [a, b]$ such that*

$$\int_a^b f(x)g(x)dx = f(z) \int_a^b g(x)dx.$$

Proof. Assume wlog that $g(x) \geq 0$ for $x \in [a, b]$. Then $f(x)$ reaches a minimum m and maximum M on $[a, b]$, so for all $x \in [a, b]$, $m \leq f(x) \leq M$, so

$$\begin{aligned}
&mg(x) \leq f(x)g(x) \leq Mg(x), \\
\to &m \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx, \\
\leftrightarrow &m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M.
\end{aligned}$$

By the intermediate value theorem, there is a $z \in [a, b]$ such that,

$$f(z) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}.$$

□

Calculus Techniques

IF YOU TAKE the derivative of a function $f(x)$ at x_0, you are looking at by how much $f(x_0)$ increases if you increase x_0 by the tiniest amount. If you do this for all values of x in an interval $[a, b]$ and add all these changes together, you end up with the difference of f between the starting and end point, i.e. $f(b) - f(a)$. This is the main intuition behind the fundamental theorem of integral calculus.

Theorem 4 (Fundamental theorem of integral calculus)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and*

$$F(x) = \int_a^x f(t)dt,$$

is an integral of $f(x)$, then the derivative of $F(x)$ is

$$\frac{d}{dx}[F(x)] = F'(x) = f(x).$$

Proof. **Case 1:** $x \in]a, b[$ (i.e. $a < x < b$): let $h \in \mathbb{R}$ be such that $x + h \in]a, b[$. By definition, we have $F(x + h) = \int_a^{x+h_t} f(y)dy$ and $F(x) = \int_a^x f(y)dy$. Taking the differences gives,

$$F(x + h) - F(x) = \int_x^{x+h} f(y)dy.$$

From the integral intermediate value theorem, there exists a number $z \in [x, x + h]$ such that,

$$F(x + h) - F(x) = \int_x^{x+h} f(y)dy = f(z)h,$$
$$\leftrightarrow \frac{F(x + h) - F(x)}{h} = f(z).$$

Now, let $h \to 0$. As $z \in [x, x + h]$, we have that get $z \to x$ along this sequence. As such, so $\lim_{h \to 0} \frac{F(x+h)-F(x)}{h}$ exists and is equal to $f(x)$. This shows that $F(x)$ is differentiable on $]a, b[$ and $F'(x) = f(x)$ for all $x \in]a, b[$.

Case 2: $x = a$. Observe that, for all $h \in \mathbb{R}_+$ small enough, by the intermediate value theorem

$$F(a + h) - F(a) = \int_a^{a+h} f(y)dy = f(z)h,$$

for some $z \in [a, a+h]$. Dividing both sides by h and taking a sequence $h \to 0$ from the right, we see that $F'(a) = f(a)$.
Case 3: $x = b$. The proof is similar taking a sequence $h \in \mathbb{R}_-$. $\square$

GIVEN THAT DIFFERENTIATION and integration are so tightly linked, it is no surprise that most manipulation rules for differentiation produce a related rule for integration. Here we discuss three of such rules.

Let y be a function of x and let f be a function of y. Then chain rule for differentiation states that, **The chain rule** for differentiation is,[2]

$$\frac{d}{dx}\left[f(y(x))\right] = f'(y)y'(x).$$

The counterpart of the chain rule for differentiation is the **change of variables** rule in integration.

[2] Of course, this is only valid if y is differentiable at x and f is differentiable at $y(x)$.

Theorem 5. *Let $h : [t_0, t_1]$ be a C^1 function with $h(t_0) = a, h(t_1) = b$ Let $f : [a, b] \to \mathbb{R}$ be a second continuous function, then*

$$\int_a^b f(x)dx = \int_{h(t_0)}^{h(t_1)} f(x)dx = \int_{t_0}^{t_1} f(h(t))h'(t)dt.$$

Proof. Define $F(t) = \int_a^x f(y)dy$, then by the fundamental theorem of integration, $F'(x) = f(x)$. By the chain rule, we have that for all $t \in [t_0, t_1]$,

$$\frac{d}{dt}F(h(t)) = F'(h(t))h'(t) = f(h(t))h'(t).$$

Integrating both sides with respect to t, gives,

$$\int_{t_0}^{t_1} f(h(t))h'(t)dt = F(h(t_1)) - F(h(t_0)) = \int_{h(t_0)}^{h(t_1)} f(x)dx = \int_a^b f(x)dx.$$

$\square$

Example: Let us solve

$$\int_0^{\pi/4} \tan(x)dx = \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}dx.$$

Introduce the change of variables $u = \cos(x)$, then

$$du = -\sin(x)dx,$$

so the integral becomes,

$$\begin{aligned}
&\int_{\cos(0)}^{\cos(\pi/4)} -\frac{1}{u}du, \\
&\int_{1}^{\cos(\pi/4)} -\frac{1}{u}du, \\
&= \int_{\cos(\pi/4)}^{1} \frac{1}{u}du, \\
&= [\ln(u)]_{\cos(\pi/4)}^{1}, \\
&= -\ln(\cos(\pi/4)).
\end{aligned}$$

A SECOND WELL known rule for differentiation is given by the **product rule**. If u and v are two differentiable functions, then,

$$\frac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x).$$

The counterpart of this rule in integration is called **integration by parts**. It easily follows by integrating both sides of the product rule (with some rearrangements).,

$$\begin{aligned}
\int_a^b u(x)v'(x)dx &= [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)dx, \\
&= u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)dx.
\end{aligned}$$

or briefly,

$$\int uv'dx = uv - \int vu'dx.$$

Example: Lets evaluate

$$\int_{x_0}^{x_1} xe^x dx,$$

Let $u(x) = x$ and $v'(x) = e^x$. Then $u'(x) = 1$ and $v(x) = e^x$. Therefore

$$\begin{aligned}
\int_{x_0}^{x_1} xe^x &= x_1e^{x_1} - x_0e^{x_0} - \int_{x_0}^{x_1} e^x dx, \\
&= (x_1 - 1)e^{x_1} - (x_0 - 1)e^{x_0}.
\end{aligned}$$

As a second exalple, let $x_0, x_1 > 0$ and consider the integral

$$\int_{x_0}^{x_1} \ln(t)dt.$$

First we perform a change of variables $u = \ln(t)$ (i.e. $t = e^u$) then $\frac{dt}{du} = e^u$ so,

$$\int_{\ln(x_0)}^{\ln(x_1)} ue^u du.$$

Evaluating as above, we get

$$\begin{aligned}
&[(u-1)e^u]_{\ln(x_0)}^{\ln(x_1)}, \\
&= (\ln(x_1) - 1)x_1 - (\ln(x_0) - 1)x_0.
\end{aligned}$$

Leibniz' rules

THE LEIBNIZ' RULES provide us with the tools to take derivative of expressions that involve integrals. As a first part of the Leibniz' rule let us show that it is (in general) possible to exchange integration and differentiation.[3]

[3] In particular, the function should be integrable, differentiable and the partial derivative should also be integrable. A sufficient condition is that the function is C^1 bounded (from above and below) and that the first derivative is also bounded.

Theorem 6. *Assume that $f(x, y)$ is defined and continuous on $[a, b] \times [c, d]$. Then,*

$$\frac{d}{dx}\left[\int_c^d f(x,y)dy\right] = \int_c^d \frac{\partial f(x,y)}{\partial x} dy.$$

Proof. Consider the expression,

$$\begin{aligned}
\frac{d}{dx}\int_c^d f(x,y)dy &= \frac{d}{dx}\left[\int_c^d \int_a^x f_x(z,y)dz\right]dy,\\
&= \frac{d}{dx}\int_a^x \left[\int_c^d f_x(z,y)dy\right]dz,\\
&= \int_c^d f_x(x,y)dy.
\end{aligned}$$

The the first equality uses the fundamental theorem of integral calculus. The second interchanges two integral signs.[4] The third line is another application of the fundamental theorem of integral calculus. □

[4] Exchanging integration signs is known as Fubini's theorem.

Let us now look at the case where not only the function f may depend on x but also the limits of integration can be functions of x.

Theorem 7. *If $f(x, t)$ is integrable and C^1 on $[a, b] \times [c, d]$ and $a(x)$ and $b(x)$ are C^1 with values in $[c, d]$, then $\int_{a(x)}^{b(x)} f(x,t)dt$ is differentiable with respect to x on $[a, b]$ and for all $x \in [a, b]$,*

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,t)dt\right] = \int_{a(x)}^{b(x)} f_x(x,t)dt + f(x,b(x))b'(x) - f(x,a(x))a'(x).$$

Proof. Let $I(x) = \int_{a(x)}^{b(x)} f(x,t)dt$. From there $I(x+h) = \int_{a(x+h)}^{b(x+h_t)} f(x+h,y)dy$. Then

$$\begin{aligned}
I(x+h) - I(x) &= \int_{a(x+h)}^{b(x+h)} f(x+h,y)dy - \int_{a(x)}^{b(x)} f(x,y)dy,\\
&= -\int_{a(x)}^{a(x+h)} f(x+h,y)dy + \int_{a(x)}^{b(x)} (f(x+h,y) - f(x,y))dy\\
&\quad + \int_{b(x)}^{b(x+h)} f(x+h,y)dy,
\end{aligned}$$

As an example, let us compute $\frac{d}{dx}\int_a^b e^{-xt}dt$. Leibniz' rule gives,

$$\frac{d}{dx}\int_a^b e^{-xt}dt = \int_a^b -te^{-xt}dt.$$

Next, integrate by parts,

$$\begin{aligned}
-\int_a^b te^{-xt}dt &= -\left([-te^{-xt}/x]_a^b - \int_a^b \frac{e^{-xt}}{x}dt\right),\\
&= -\left(\frac{-be^{-xb}+ae^{-xa}}{x} - [-e^{-xt}/x^2]_a^b\right),\\
&= \frac{be^{-xb}-ae^{-xa}}{x} + \frac{-e^{-xb}+e^{-xa}}{x^2}.
\end{aligned}$$

As a second example, let us compute

$$\frac{d}{dx}\int_0^{2x} te^x dt.$$

From Leibniz' rule,

$$\begin{aligned}
\frac{d}{dx}\int_0^{2x} te^x dt &= \int_0^{2x} te^x dt + 4xe^x,\\
&= [t^2/2]_0^{2x} e^x + 4xe^x,\\
&= (2x^2+4x)e^x.
\end{aligned}$$

By the intermediate value theorem, there is a $z_1 \in [a(x), a(x+h)]$ such that,

$$-\int_{a(x)}^{a(x+h)} f(x+h, y)dy = -f(x+h, z_1)[a(x+h) - a(x)],$$

and there is a $z_2 \in [b(x), b(x+h)]$ such that,

$$\int_{b(x)}^{b(x+h)} f(x+h, y)dy = f(x+h, z_2)[b(x+h) - b(x)].$$

Then,

$$\begin{aligned}\frac{I(x+h) - I(x)}{h} =& \frac{\int_{a(x)}^{b(x)} (f(x+h, y) - f(x, y))dy}{h} - f(x+h, z_1)\frac{[a(x+h) - a(x)]}{h}, \\ &+ f(x+h, z_2)\frac{[b(x+h) - b(x)]}{h}.\end{aligned}$$

Taking a sequence $h \to 0$ (and therefore $z_1 \to a(x)$ and $z_2 \to b(x)$, and interchanging differentiation and integration, the right hand side becomes

$$\int_{a(x)}^{b(x)} f_x(x, t)dt - f(x, a(x))a'(x) + f(x, b(x))b'(x).$$

□

Concave and convex functions

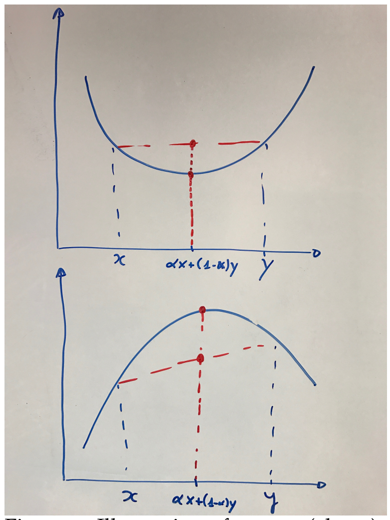

Figure 3: Illustration of convex (above) and concave (below) functions. Convex functions are happy , concave functions are sad.

CONCAVITY AND CONVEXITY play a fundamental part in optimization theory as they have a unique maximum (or minimum). In this part, we give three equivalent definitions of concavity and convexity.

The first definition, is the one we all know, a function $f : S \to \mathbb{R}^n$ where $S \subseteq \mathbb{R}^n$ is a convex set is called **concave** if for all vectors $\mathbf{x}, \mathbf{y} \in S$ and all numbers $\alpha \in [0, 1]$,

$$f(\alpha\mathbf{x} + (1-\alpha)y) \geq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}).$$

If the inequality is reversed, the function f is called **convex**. If the inequality is strict for all $\mathbf{x} \neq \mathbf{y}$ and $\alpha \in]0, 1[$ then the function is called **strictly concave (convex)**. The following theorem gives an equivalent definition of concavity (convexity) in cases the function f is C^1. It is very useful in the proof of many theoretical results.

Theorem 8. *A C^1 function f on a convex set $S \subseteq \mathbb{R}^n$ is concave if and only if for all $\mathbf{x}, \mathbf{y} \in S$,*

$$f(\mathbf{x}) - f(\mathbf{y}) \leq \sum_i \frac{\partial f(\mathbf{y})}{\partial y_i}(x_i - y_i) = \nabla_y f(\mathbf{y})(\mathbf{x} - \mathbf{y}).$$

The function f is convex if and only if for all $\mathbf{x}, \mathbf{y} \in S$,

$$f(\mathbf{x}) - f(\mathbf{y}) \geq \sum_i \frac{\partial f(\mathbf{y})}{\partial y_i}(x_i - y_i) = \nabla_y f(\mathbf{y})(\mathbf{x} - \mathbf{y}).$$

Proof. We proof the case of concavity. The proof for convexity of f is similar. ($\rightarrow$) Let $\mathbf{x}, \mathbf{y} \in S$, then for all $\alpha \in]0,1[$,

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) = \alpha\left[f(\mathbf{x}) - f(\mathbf{y})\right] + f(\mathbf{y}),$$
$$\leftrightarrow f(\mathbf{x}) - f(\mathbf{y}) \leq \frac{f(\mathbf{y} + \alpha(\mathbf{x} - \mathbf{y})) - f(\mathbf{y})}{\alpha}.$$

Define the function $g(\alpha) = f(\mathbf{y} + \alpha(\mathbf{x} - \mathbf{y}))$,[5] then we have that for all $\alpha \in]0,1[$,

$$f(\mathbf{x}) - f(\mathbf{y}) \leq \frac{g(\alpha) - g(0)}{\alpha}.$$

Taking the limit for $\alpha \rightarrow 0$ gives,

$$f(\mathbf{x}) - f(\mathbf{y}) \leq g'(0) = \sum_i \frac{\partial f(\mathbf{y})}{\partial y_i}(x_i - y_i),$$

as was to be shown.
($\leftarrow$) Assume the inequalities are satisfied, then we have that, for all $\mathbf{x}, \mathbf{y} \in S$,

$$f(\mathbf{x}) - f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \sum_i f_i(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})(1-\alpha)(x_i - y_i),$$
$$f(\mathbf{y}) - f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \sum_i f_i(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})\alpha(y_i - x_i).$$

Multiplying left hand right hand side of the first equation by α and of the second equation by $(1-\alpha)$ and adding the two equations together gives $f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})$ as desired. □

[5] Observe that here we keep $\mathbf{y}$ and $\mathbf{x}$ fixed. So g is a function of one variable, α.

LET A BE a symmetric $n \times n$ matrix.[6] We say that A is **negative semi-definite** if for all vectors $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{z}'A\mathbf{z} \leq 0$ or equivalently,

$$\sum_i \sum_j z_i z_j a_{i,j} \leq 0.$$

The matrix A is **negative definite** if for all nonzero $\mathbf{z}$, $\mathbf{z}'A\mathbf{z} < 0$, it is **positive semidefinite** if for all $\mathbf{z} : \mathbf{z}'A\mathbf{z} \geq 0$ and **positive definite** if the inequality is strict for all $z \neq 0$.

If $f : S \rightarrow \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ is a C^2 function, then the **Hessian** of f at the point $\mathbf{x}^* \in S$ is the $n \times n$ symmetric matrix of second order partial derivatives,[7]

$$H_f(\mathbf{x}^*) = \begin{bmatrix} f_{11}(\mathbf{x}^*) & f_{12}(\mathbf{x}^*) & \dots & f_{1n}(x^*) \\ f_{21}(\mathbf{x}^*) & f_{22}(\mathbf{x}^*) & \dots & f_{2,n}(x^*) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\mathbf{x}^*) & f_{n2}(\mathbf{x}^*) & \dots & f_{nn}(\mathbf{x}^*) \end{bmatrix}$$

[6] A matrix A is symmetric if $a_{i,j} = a_{j,i}$ where $a_{i,j}$ is the element on row i and column j. In other words, A is symmetric if it is equal to it's transpose. Basically, we can reflect A on its diagonal and obtain the same matrix A again.

[7] Here we use the notation,

$$f_{i,j}(\mathbf{x}^*) = \frac{\partial^2 f(\mathbf{x}^*)}{\partial x_i \partial x_j}.$$

The following theorem shows that convexity or concavity can be verified by looking at the definiteness of the Hessian.

Theorem 9. *The C^2 function f is concave on the open set S if and only if $H_f(\mathbf{x})$ is negative semidefinite for all $\mathbf{x} \in S$. The function f is convex if and only if the Hessians are positive semidefinite.* [8]

If the Hessians are negative (positive) definite, then the function f is strictly concave (convex).

[8] Observe that if f is a one-dimensional function (i.e. $S \subseteq \mathbb{R}$), then the Hessian is negative semi-definite if and only if the second order derivative $f''(x) \leq 0$ for all $x \in S$. So concavity of f is equal to $f''(x) \leq 0$, but we already knew this.

Proof. Assume f is concave, then for all $\mathbf{x}$ and $\mathbf{z}$ and all numbers $t > 0$ small enough.

$$f(\mathbf{x}) - f(\mathbf{x} + t\mathbf{z}) \leq \sum_i \frac{\partial f(\mathbf{x} + t\mathbf{z})}{\partial x_i}(-tz_i),$$

$$f(\mathbf{x} + t\mathbf{z}) - f(\mathbf{x}) \leq \sum_i \frac{\partial f(\mathbf{x})}{\partial x_i}(tz_i).$$

Adding the left and right hand sides together gives,

$$0 \leq \sum_i \left(\frac{\partial f(\mathbf{x})}{\partial x_i} - \frac{\partial f(\mathbf{x} + t\mathbf{z})}{\partial x_i} \right) tz_i.$$

Define $g_i(t) = \frac{\partial f(\mathbf{x}+t\mathbf{z})}{\partial x_i}$.[9] Then,

[9] Here $\mathbf{x}$ and $\mathbf{z}$ are fixed, so g_i is a function of t alone.

$$0 \leq \sum_i (g_i(0) - g_i(t))\, tz_i.$$

$$\leftrightarrow 0 \leq -\sum_i \frac{g_i(t) - g_i(0)}{t} z_i.$$

Taking the limit for $t \to 0$ gives,[10]

[10] This uses the fact that $g_i'(0) = \sum_j f_{i,j} z_j$. You should check this.

$$0 \leq -\sum_i g_i'(0) z_i,$$

$$\leftrightarrow 0 \leq -\sum_i \sum_j f_{ij}(\mathbf{x}) z_j z_i = -\mathbf{z}' H_f(\mathbf{x}) \mathbf{z}.$$

Given that the choice of $\mathbf{x} \in S$ and $\mathbf{z} \in \mathbb{R}^n$ was arbitrary, we have that $H_f(\mathbf{x})$ is negative semi-definite for all $\mathbf{x} \in S$.

For the reverse, assume that $H_f(\mathbf{x})$ is negative semi-definite. Let $\mathbf{x}, \mathbf{y} \in S$ and let $\mathbf{z} = (\mathbf{y} - \mathbf{x})/\alpha$. Then $\mathbf{y} = \mathbf{x} + \alpha\mathbf{z} \in S$. Define $g(\alpha) = f(\mathbf{x} + \alpha\mathbf{z})$ and take a second order Taylor expansion of $g(\alpha)$ around $g(0)$. Then

$$g(\alpha) = g(0) + g'(0)\alpha + \frac{\alpha^2}{2} g''(\beta),$$

for some $\beta \in [0, \alpha]$. Then,

$$f(\mathbf{x} + \alpha\mathbf{z}) = f(\mathbf{x}) + \alpha \sum_i \frac{\partial f(\mathbf{x})}{\partial x_i} z_i + \frac{\alpha^2}{2} \sum_i \sum_j f_{ij}(\mathbf{x} + \beta\mathbf{z}) z_i z_j.$$

By negative semi-definiteness, the last term is smaller or equal to zero. So,

$$\begin{aligned} f(\mathbf{x}+\alpha\mathbf{z}) - f(\mathbf{x}) &\leq \sum_i \frac{\partial f(\mathbf{x})}{\partial x_i}(\mathbf{x}+\alpha\mathbf{z}-\mathbf{x}) \\ \leftrightarrow f(\mathbf{y}) - f(\mathbf{x}) \leq \sum_i \frac{\partial f(\mathbf{x})}{\partial x_i}(\mathbf{y}-\mathbf{x}). \end{aligned}$$

so f is concave. □

Now, how do we check if a symmetric matrix A is positive or negative (semi)-definite. There is an 'easy' rule depending on leading minors. The matrix H_k is the kth **leading principal minor** of the matrix of the matrix A if it is equal to the matrix obtained from A by deleting all but the first k rows and columns. The matrix R_k is **a principal minor** of A if it is obtained by deleting $n-k$ rows and the corresponding $n-k$ columns of A. Then,[11]

- A is positive definite if and only if $\det(H_k) > 0$ for all $k \leq n$.
- A is negative definite if and only if $(-1)^k \det(H_k) > 0$ for all $k \leq n$.
- A is positive definite if and only if $\det(R_k) \geq 0$ for all principal minors.
- A is negative semi-definite if and only if $(-1)^k \det(R_k) \geq 0$ for all principal minors.

[11] A proof of this is beyond the scope of these notes.

Free or unconstrained optimization

Unconstrained optimization problems try to find the maximal or minimal value some (mostly continuous) function without imposing any additional restrictions on the values of the underlying variables.[12] We first have a look at single valued functions and then consider functions of more than one variable.

[12] Of course, the values should ly in the domain of the function.

Definition 1. *Let $f : [a,b] \to \mathbb{R}$ be a C^1 function where $a, b \in \mathbb{R}$ and $a < b$. If $f(x^*) \geq f(x)$ for all $x \in S$, then x^* is a* ***global maximum****. If $f(x^*) > f(x)$ for all $x \in S$, then x^* is a* ***strict global maximum****.*

If for all x in a small neighbourhood of x^*, e.g. $x \in [x^* - \varepsilon, x^* + \varepsilon] \cap S$ for some $\varepsilon > 0$, $f(x^*) \geq f(x)$ then x^* is called a ***local maximum****. If $f(x^*) > f(x)$, then x^* is a* ***strict local maximum****.*

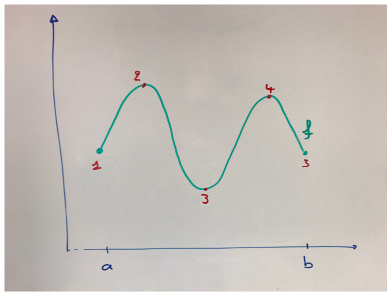

Figure 4: points 1, 3 and 5 are local minima and 2 and 4 are local maxima. Point 2 is the global maximum while point 3 is the global minimum.

Assume that x^* is a local maximum. Then there are three possibilities:
Case 1: $x = a$ In this case, we have that for some $\varepsilon >$ and all $x \in [a, a+\varepsilon]$, there is a $z \in [a, a+\varepsilon]$[13]

[13] The existence of z is guaranteed by the integrated mean value theorem: as $f = \int_a^x f'(y)dy$, there is a $z \in [a,x]$ such that

$$\begin{aligned} f'(z)(x-a) &= \int_a^x f'(y)dy, \\ &= f(x) - f(x). \end{aligned}$$

$$\begin{aligned} 0 &\geq f(x) - f(a) = f'(z)(x-a), \\ \leftrightarrow 0 &\geq \frac{f(x)-f(a)}{x-a} = f'(z). \end{aligned}$$

Now, take a sequence $x \to a$. Then we see that $f'(a) \leq 0$.
Case 2: $x = b$ In this case, we have that for all $x \in [b-\varepsilon, b]$, there is a $z \in [x, b]$ such that,

$$\begin{aligned} 0 &\leq f(b) - f(x) = f'(z)(b-x), \\ \leftrightarrow 0 &\leq \frac{f(b)-f(x)}{b-x} = f'(z). \end{aligned}$$

Taking a sequence $x \to b$, shows that $f'(b) \geq 0$.
Case 3: $a < x^* < b$ In this case, we have that for all $x \in [x^* - \varepsilon, x^* + \varepsilon]$ there is a z between x^* and x such that,

$$0 \leq f(x^*) - f(x) = f(z)(x^* - x).$$

As such, for $x^* > x$, we have that $f'(z) > 0$ and for $x^* < x$, we have that $f'(z) < 0$. By continuity of f', and letting $\varepsilon \to 0$, it follows that $f'(x^*) = 0$, i.e. the derivative at the local optimum is equal to zero. This is the well known first order condition for interior solutions.

Summarizing, we have that,

1. If a is a local optimum $\to f'(a) \leq 0$,
2. If b is a local maximum $\to f'(b) \geq 0$,
3. If x is an interior local maximum $\to f'(x) = 0$.

These conditions are called the **first order conditions**. In general they are not sufficient. One important exception is when f is concave. Indeed, if f is concave and $f'(a) \leq 0$ then for all $x \in [a, b]$,[14]

$$f(x) - f(a) \leq f'(a)(x-a) \leq 0,$$

so a is also a global maximum. If $f'(b) \geq 0$ then for all $x \in [a, b]$,

$$f(x) - f(b) \leq f'(b)(x-b) \leq 0,$$

so b is a global maximum. Finally, if $f'(x^*) = 0$ then, for all $x \in [a, b]$.

$$f(x) - f(x^*) = f'(x^*)(x - x^*) = 0,$$

so again x^* is a global maximum.

Let us now have a look at maximization problems where f is a function of many, say n, variables $x_1, \ldots, x_n$ and f is defined on a set $S \subseteq \mathbb{R}^n$. If the point $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ belongs to S and if

[14] Notice that $(x-a) \geq 0$ for all $x \in [a, b]$.

$f(\mathbf{x}^*)$ is greater or equal than the values of f at all other points $\mathbf{x} = (x_1, \ldots, x_n)$ of S, so

$$\forall \mathbf{x} \in S : f(\mathbf{x}^*) \geq f(\mathbf{x}).$$

then $\mathbf{x}^*$ is called a **global maximum** point for f in S and $f(\mathbf{x}^*)$ is called the maximum value. If the inequality is strict for all $\mathbf{x} \neq \mathbf{x}^*$ and $\mathbf{x} \in S$, then $\mathbf{x}^*$ is a **strict maximum** point of f in S. The (strict) minimum point is defined by reversing the inequality sign. Maxima and minima are collected using the term extreme points. Local maxima and minima are points that are maximal and minimal in a small neighborhood around the point.[15]

A **stationary point** of f is a point where all the first-order partial derivatives are 0.[16]

[15] By the extreme value theorem, any continuous function $f : S \to \mathbb{R}$ defined on a compact set $S \subset \mathbb{R}^n$ always attains a global maximum and global minimum on that set. This maximum can either be interior or at a corner of the domain.

[16] So for all x_i, $\frac{\partial f(\mathbf{x})}{\partial x_i} = 0$.

Theorem 10. *Let f be defined on a set $S \subseteq \mathbb{R}^n$ and let $\mathbf{x}^*$ be an interior point in S (i.e. there is a $\varepsilon > 0$ such that $B_\varepsilon(x^*) \subseteq S$) at which f has partial derivatives. A necessary condition for $\mathbf{x}^*$ to be a maximum or minimum point for f is that $\mathbf{x}^*$ is a stationary point for f, i.e.*

$$\forall i = 1, \ldots, n : \frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0.$$

Proof. For $i = 1, \ldots, n$, consider the functions,[17]

$$g_i(x) = f(x_1^*, \ldots, x_{i-1}^*, x, x_{i+1}^*, \ldots, x_n^*).$$

[17] Here we keep all values $x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_n^*$ fixed and only vary x_i.

The function g_i reaches a local maximum at the interior point x_j^* from the previous analysis,

$$g^{i\prime}(x_i^*) = f_i(\mathbf{x}^*) = 0.$$

This holds for all $i \leq n$ so all partial derivatives should be equal to zero. □

Theorem 11. *Let f be defined in a convex set $S \subseteq \mathbb{R}^n$ and let $\mathbf{x}^*$ be an interior point of S. Assume also that f is C^1 in an open ball around $\mathbf{x}^*$.*

- *If f is concave, then $\mathbf{x}^*$ is a global maximum if and only if $\mathbf{x}^*$ is a stationary point.*
- *If f is convex, then $\mathbf{x}^*$ is a global minimum if and only if $\mathbf{x}^*$ is a stationary point.*

Proof. The only if part follows from the previous theorem. Now assume that $\mathbf{x}^*$ is a stationary point and that f is concave. Then as f is concave, we have that for all $x \in S$,

$$f(\mathbf{x}) - f(\mathbf{x}^*) \leq \sum_i \frac{\partial f(\mathbf{x})}{\partial x_i}(x_i - x_i^*) = 0,$$

which means that $f(x^*)$ is a global maximum. □

Constrained Optimization

In a constrained optimization problem, we are optimizing a function of some variables, but the possible values of these variables have to satisfy some constraints. The implicit value theorem is a key tool in the analysis of these problems.

Theorem 12 (Implicit function theorem)**.** *Consider a C^1 function of two variables $g(x,y)$. If at (x_0,y_0), $g(x_0,y_0) = 0$ while $\frac{\partial g(x_0,y_0)}{\partial y} \neq 0$. Then there is a region $[x_1,x_2] \times [y_1,y_2]$ containing (x_0,y_0) such that for all $x \in [x_1,x_2]$ there is exactly one value $y = \psi(x)$ such that $\psi(x) \in [y_1,y_2]$ and $g(x,\psi(x)) = 0$. In addition, the function ψ is continuous and differentiable with*

$$\psi'(x_0) = -\frac{\frac{\partial g(x_0,y_0)}{\partial x}}{\frac{\partial g(x_0,y_0)}{\partial y}}.$$

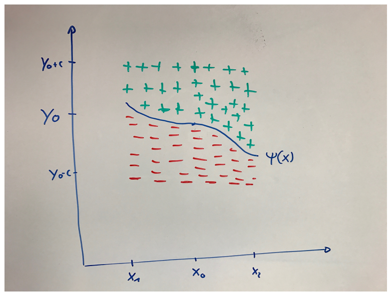

Figure 5: Illustration of the proof of the implicit function theorem: green where $f(x,y) > 0$ and red where $f(x,y) < 0$. as $f(x,y)$ is increasing in y we have that for each x, there is only a single point $\psi(x)$ where $f(x,\psi(x)) = 0$.

Proof. Assume without loss of generality that $\frac{\partial g(x_0,y_0)}{\partial y} > 0$. Given that $g(x_0,y_0) = 0$, we have that that $g(x_0,y_0)$ is locally strictly increasing in its second coordinate. So, there is a $c > 0$ such that $g(x_0,y_0+c) > 0 > g(x_0,y_0-c)$. Also because of continuity of f, there is an interval $[x_1,x_2]$ such that for all $x \in [x_1,x_2]$, $g(x,y_0+c) > 0 > g(x,y_0-c)$. By the intermediate value theorem, for all $x \in [x_1,x_2]$ there is an $y \in [y_0-c,y_0+c]$ such that $g(x,y) = 0$. Let us show that there is only one such value so that y is a function of x.

If not, there are values $\tilde{y} > \hat{y}$ such that $g(x,\tilde{y}) = g(x,\hat{y}) = 0$. But then,

$$0 = g(x,\tilde{y}) = g(x,\hat{y}) + \int_{\hat{y}}^{\tilde{y}} g_y(x,y)dy > g(x,\hat{y}) = 0,$$

a contradiction.

Given this, there is a function $\psi(x)$ such that $g(x,\psi(x)) = 0$ for all x in $[x_1,x_2]$. As g is C^1 we can take the derivative and obtain,

$$\frac{\partial g(x_0,y_0)}{\partial x} + \frac{\partial g(x_0,y_0)}{\partial y}\psi'(x_0) = 0,$$

$$\leftrightarrow \psi'(x_0) = -\frac{\frac{\partial g(x_0,y_0)}{\partial x}}{\frac{\partial g(x_0,y_0)}{\partial y}}.$$

□

Now that we are armed with the implicit function theorem let us try to tackle the following maximization problem,

$$\max_{x,y} f(x,y),$$

$$\text{s.t. } g(x,y) = c.$$

where $f(x,y)$ and $g(x,y)$ are two C^1 functions of two variables. Assume that (x^*,y^*) provide a solution to this problem and assume that

$$\frac{\partial g(x^*,y^*)}{\partial y} \neq 0.$$

By the implicit function theorem (see above), we know that there is a function $\psi(x)$, defined locally around x^*, such that $g(x,\psi(x)) = 0$.[18]

Let us substitute $\psi(x)$ into the objective function,

$$\max_x f(x,\psi(x)).$$

Now for each of the values x in this small neighbourhood of x^*, we know that $g(x,\psi(x)) = 0$, so $(x,\psi(x))$ satisfies the constraint of the optimization problem. However, only x^* is the optimal one. As such, we can use the first order condition on $f(x,\psi(x))$ and obtain,[19]

$$\begin{aligned} f_x(x^*,\psi(x^*)) + f_y(x^*,\psi(x^*))\psi'(x^*) = 0,\\ \leftrightarrow f_x(x^*,y^*) - f_y(x^*,y^*)\frac{g_x(x^*,y^*)}{g_y(x^*,y^*)} = 0.\end{aligned}$$

Define $\lambda^* = \frac{f_y(x^*,y^*)}{g_y(x^*,y^*)}$. Then we can write,

$$\begin{aligned} f_x(x^*,y^*) - \lambda^* g_x(x^*,y^*) = 0,\\ f_y(x^*,y^*) - \lambda^* g_y(x^*,y^*) = 0.\end{aligned}$$

These conditions can be summarized by using the Lagrange function.

$$L(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c).$$

Then the above two conditions give,

$$\begin{aligned} &L_x(x^*,y^*;\lambda^*) = 0,\\ &L_y(x^*,y^*;\lambda^*) = 0,\\ &L_\lambda(x^*,y^*;\lambda^*) = g(x^*,y^*) - c = 0.\end{aligned}$$

These are the well known Lagrangian first order conditions for an interior solution.

THE LAGRANGE MULTIPLIER λ has a convenient interpretation. Let $L(c)$ be the optimal value for the following optimization problem,

$$L(c) = \max_{x,y} f(x,y) \text{ s.t. } g(x,y) = c.$$

Let $x(c), y(c)$ be the solution of this maximization problem.[20] For different values of c, there will be different solutions, so the optimal value and optimal solutions will be a function of the value of c (parameter).

$$L(c) = f(x(c),y(c)) = \max\{f(x,y) : g(x,y) = c\}.$$

[18] And

$$\psi'(x_0) = -\frac{\frac{\partial g(x_0,y_0)}{\partial x}}{\frac{\partial g(x_0,y_0)}{\partial y}}.$$

[19] Here, for ease of notation, we use f_x, f_y, g_x and g_y to indicate the partial derivatives of f and g with respect to x and y.

[20] $L(c)$ is obtained by plugging in the solutions of the optimization problem into the objective function: $L(c) = f(x(c), y(c))$.

Then, taking the total derivative with respect to c,[21] gives,

$$\frac{d}{dc}L(c) = f_x(x(c), y(c))x'(c) + f_y(x(c), y(c))y'(c),$$
$$= \lambda^* \left[g_x(x(c), y(c))x'(c) + g_y(x(c), y(c))y'(c)\right].$$

[21] Assuming that $x(c)$ and $y(c)$ are differentiable with respect to c

The equality is obtained by using the Lagrangian first order conditions. Now, as $g(x^*(c), y^*(c)) = c$ we also have that,[22]

[22] Take derivatives with respect to c on both sides.

$$g_x(x(c), y(c))x'(c) + g_y(x(c), y(c))y'(c) = 1.$$

As such,

$$\frac{d}{dc}L(c) = \lambda^*$$

The intuition is now clear, the Lagrange multiplier λ^* measures by how much the optimal value of the maximization problem changes if the constrained c changes by an infinitesimal amount. It is therefore the marginal value of the constraint. Economists often use the term 'shadow price' to denote this quantity.[23].

[23] The shadow price is the implicit price you would be willing to 'pay' (in terms of the objective function) in order to relax the constraint by one unit

SIMILAR TO THE setting without constraints, constraint optimization problems also have a set of sufficient second order conditions. This second order condition is that the function $F(x) = f(x, \psi(x))$ is (locally) concave at x^*, i.e. the second derivative $F''(x^*) < 0$. Now, the first derivative reads,

$$F_x(x^*) = f_x(x^*, y^*) + f_y(x^*, y^*)\psi'(x^*),$$
$$\leftrightarrow F_x(x^*) = L_x(x^*, y^*; \lambda^*) + L_y(x^*, y^*; \lambda^*)\psi'(x^*).$$

This equality follows from the fact that $g_x(x^*, y^*) + g_y(x^*, y^*)\psi'(x^*) = 0$.[24] Again differentiating both sides gives,

[24] So we can add this term to the right hand side and keep the equality for all x.

$$F_{x,x}(x^*) = L_{x,x} + 2L_{x,y}\psi' + L_{y,y}\left[\psi'\right]^2 + L_y\psi''.$$

The last term is zero from the first order conditions.[25] Then, substituting $\psi' = -g_x/g_y$, gives,

[25] $L_y = f_y + \lambda g_y = 0$.

$$F_{x,x} = \frac{1}{\left[g_y\right]^2}\left[\ \left[g_y\right]^2 L_{x,x} - 2L_{x,y}g_x g_y + L_{y,y}\left[g_x\right]^2\ \right].$$

The expression between brackets is the exact negative of the determinant of the "bordered Hessian"[26]

$$\begin{bmatrix} L_{\lambda\lambda} = 0 & L_{\lambda x} = -g_x & L_{\lambda y} = -g_y \\ L_{x\lambda} = -g_x & L_{xx} & L_{xy} \\ L_{y\lambda} = -g_y & L_{yx} & L_{yy} \end{bmatrix}_{(x^*, y^*, \lambda^*)}$$

[26] Remember the determiniant of a 3 by 3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is given by

$$aci + bfg + dhc - ceg - bdi - fha.$$

So if this determinant is positive, then $F''(x^*) < 0$ and the optimum is a maximum.

MOST OF WHAT we established above can be generalized to problems with more than two variables. Consider the objective function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and a number of constraint functions $g^j : D \to \mathbb{R}, j = 1, \dots, m$ with values $c^j \in \mathbb{R}$.

Theorem 13. *If f and g_j are C^1 in the neighbourhood of an interior point $x^* \in D$ and the rank of $J_g(x^*)$ (the Jacobian of the g_j's) at x^* equals m, if a maximum or a minimum of f subject to the constraints $g^j(x) = c^j$ occurs at x^* and if,*

$$L(x, \lambda) = f(x) - \sum_j \lambda^j (g^j(x) - c^j),$$

then there is a $\lambda^* \in \mathbb{R}^m$ such that for all $i = 1, \dots, n$,

$$L_{x_i}(x^*, \lambda^*) = 0.$$

and $L_{\lambda^j}(x^*, \lambda^*) = 0$.

The proof of this result is very similar to the proof of the single constraint optimization problem. Except here we have a multidimensional implicit function theorem. There are also a set of sufficient second order conditions. If f and g_j are C^2 in the neighbourhood of x^*, an interior point of D at which Jacobian J_g is of rank m and such that if $L(x, \lambda) = f(x) - \sum_j \lambda_j (g_j(x) - c_j)$, there exists $\lambda^* \in \mathbb{R}^m$ such that for all $i = 1, \dots, n$, $\frac{\partial L}{\partial x_i}(x^*, \lambda^*) = 0$ and for all $j = 1, \dots, m$, $\frac{\partial L}{\partial \lambda_j}(x^*, \lambda^*) = 0$. Then if

$$\bar{H}(x^*, \lambda^a st) = \left[\begin{array}{ccc|ccc} & & & -g^1_{x_1} & \cdots & -g^1_{x_n} \\ & 0 & & \vdots & \ddots & \vdots \\ & & & -g^m_{x_1} & \cdots & -g^m_{x_n} \\ -g^1_{x_1} & \cdots & -g^m_{x_1} & & & \\ \vdots & \ddots & \vdots & & L_{x_i x_j} & \\ -g^1_{x_n} & \cdots & -g^m_{x_n} & & & \end{array}\right]_{(x^*, \lambda^*)}$$

- If the $(n - m)$ last principal diagonal minors alternate in sign and if $|\bar{H}(x^*, \lambda^*)|$ has the same sign as $(-1)^n$ then f attains a local maximum subject to constraints $g_j(x) - c_j = 0$ at x^*.
- If those minors all have the sign of $(-1)^n$, there is a constrained minimum at x^*.

LET US NOW consider a setting where we want to maximize a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and we have constraint functions $g^j : D \to \mathbb{R}$ for $j = 1, \dots, m$. Let $c^j \in \mathbb{R}$ but we don't impose equalities but instead inequalities on the constraints,

$$\begin{aligned}
&\max_x f(x),\\
&\text{s.t. } g^j(x) \leq c^j,\ (j = 1, \ldots, m)\\
&\quad x_i \geq 0,\ (i = 1, \ldots, n).
\end{aligned}$$

Because of the inequalities, the usual Lagrange conditions are no longer applicable.

In order to solve this problem, the key insight is to rephrase it as a problem with equality constraints. Towards this end, let us introduce for each constraint j a variable $y_j, (j = 1, \ldots, m)$ and for each variable x_i a new variable $z_i, (i = 1, \ldots, n)$ and consider the following problem,

$$\begin{aligned}
&\max_x f(x),\\
&\text{s.t. } g^j(x) + y_j^2 = c^j,\ (j = 1, \ldots, m)\\
&\quad x_i - z_i^2 = 0,\ (i = 1, \ldots, n).
\end{aligned}$$

Verify for yourself that any solution to this problem is also a solution to the problem with inequality constraints and vice versa. Given that we have a regular problem with equality constraints, we can setup the Lagrangian,

$$L(x, y, z; \lambda, \mu) = f(x) + \sum_j \lambda^j (c^j - g^j(x) - y_j^2) + \sum_i \mu^i (x_i - z_i^2).$$

In addition to the first order conditions for x_i and λ^j, we also need to take into account the first order conditions for y_j, z_i and μ^i. The first order conditions are given by,

$$\begin{aligned}
L_{x_i} &= f_{x_i}(x^*) - \sum_j \lambda^{j*} g^j_{x_i}(x^*) + \mu^{i*} = 0,\\
L_{y_j} &= 2\lambda^{j*} y_j = 0,\\
L_{z_i} &= -2\mu^{i*} z_i = 0,\\
L_{\lambda^j} &= c^j - g^j(x^*) - y_j^2 = 0,\\
L_{\mu^i} &= x_i - z_i^2 = 0.
\end{aligned}$$

The third condition is satisfied only if $\mu^i = 0$ or $z_i = 0$. If $z_i = 0$ then the last constraint tells us that $x_i = 0$. If the former is the case, then,

$$f_{x_i}(x^*) - \sum_j \lambda^{j*} g^j_{x_i} = 0.$$

These two conditions, together require that,

$$x_i^* \left[f_{x_i}(x^*) - \sum_j \lambda^{j*} g^j_{x_i}(x^*) \right] = 0.$$

For the second first order condition to be satisfied, it must be that either $\lambda^{j*} = 0$ or $y_j = 0$. If the latter is the case, then the second to last constraint tells us that $g^j(x^*) - c^j = 0$. As such, we obtain the condition that,

$$\lambda^{j*}\left[g^j(x^*) - c^j\right] = 0.$$

In addition, using the same reasoning as before, we see that λ^j is the shadow price of c^j and μ^i is the shadow price of allowing x_j to be slightly negative. As such, we should have that $\lambda^j, \mu^i \geq 0$.

This latter condition is called the **complementary slackness conditions**. They have intuitive interpretations. Remember that λ^j measures the marginal value of the j-th constraint.[27] Then if $\lambda_j = 0$ this means that relaxing the constraint does not increase the optimal value. This can only be if the constraint is not binding. On the other hand, if $\lambda_j > 0$, then the constraint must be binding, i.e. $g^j(x^*) = c^j$. Otherwise we could increase the optimal value without violating any constraints. In addition, the optimal value cannot decrease if the constrained is relaxed, which gives the additional condition $\lambda_j^* \geq 0$.

[27] By how much does the optimal value function increases if we relax the jth constraint by one.

Summarizing, we obtain the following conditions,[28]

[28] These conditions are called the Kuhn-Tucker first order conditions.

$$
\begin{aligned}
&L_{x_i}(x^*, \lambda^*) = f_{x_i}(\mathbf{x}^*) - \sum_j \lambda^{j*} g^j_{x_i}(x^*) \leq 0,\\
&x_i^* \geq 0,\\
&x_i^* L_{x_i}(x^*, \lambda^*) = 0,\\
&L_{\lambda^j} = g^j(\mathbf{x}^*) - c^j \geq 0,\\
&\lambda_j^* \geq 0,\\
&\lambda^{j*} L_{\lambda^j}(x^*, \lambda^*) = \lambda^{j*}[g^j(x^*) - c^j] = 0.
\end{aligned}
$$

IT IS RATHER DIFFICULT to obtain locally sufficient conditions. However, if the functions

- f and g^j's are differentiable on $D \cap \{x \in \mathbb{R}^n : x \geq 0\} = E$
- f is concave on E
- the functions g^j are convex on E,
- x^* satisfies the conditions of the Kuhn-Tucker,

then f subject to $g^j(x) \leq c^j$ and $x \geq 0$ attains a global maximum at x^*.[29]

[29] The proof is quite easy. As the g_j's are convex, the sets $E_j = \{x \in \mathbb{R}^n : g^j(x) \leq c^j\}$ are also convex. Then $\cap E_j$ is also convex and closed. As f is a concave on a convex set, it follows that any local optimum is also a global optimum.

Envelope theorems

We are already familiar with the fact that the Lagrange multiplier can be seen as the marginal value of relaxing the constraint in the optimization problem. This result is a particular case of a more general result, called the envelope theorem.[30] In any optimization problem (constrained or not), if one or several parameters are present, the envelope theorem tells you how much the optimal value of the objective function changes when one of the parameters is modified?

[30] The envelope theorem obtains its name from the fact that the optimal value function can be seen as the upper (or lower) envelope of the objective function for varying values of the parameter.

Let us first take the case of an optimization problem with no side constraints. Let $f(x,\alpha)$ be C^1 for $x \in D$, an open subset of $\mathbb{R}^n$, and a set of parameters $\alpha \in \mathbb{R}^s$. For each α the following problem is considered:

$$\max_{x\in D} f(x,\alpha).$$

Theorem 14. *Assume $x(\alpha)$ is an interior solution of the latter problem and $x(\alpha)$ is C^1 in α, then,*

$$\frac{d}{d\alpha_s}\left[f(x(\alpha),\alpha)\right] = f_{\alpha_s}(x(\alpha),\alpha).$$

Proof. Let $V(\alpha) = f(x^*(\alpha),\alpha)$. This function is C^1. then

$$\begin{aligned} V_{\alpha_s}(\alpha) &= \sum_i f_{x_i}(x(\alpha),\alpha)\frac{\partial x_i(\alpha)}{\partial \alpha_s} + f_{\alpha_s}(x(\alpha),\alpha), \\ &= f_{\alpha_s}(x(\alpha),\alpha). \end{aligned}$$

□

Let p be the price of an output y, let x be a vector of inputs and let w be the vector of input prices. Then the profit of a firm with production function $y = f(x)$ is given by,

$$\pi(p,w) = \max_x pf(x) - \sum_i w_i x_i.$$

By the envelope theorem, we have that $\pi_p(p,w) = f(x^*) = y^*$ and $\pi_{w_i}(p,w) = -x_i^*$. These are called Hotelling's lemma.

Consider now the case of an optimization problem with constraints. Let f and $g^j (j = 1,\dots,m)$ be C^1 functions on D, an open subset of $\mathbb{R}^m$, $\alpha \in \mathbb{R}^S$ and $c \in \mathbb{R}^m$. Assume $x^*(\alpha)$ is a solution ot the problem:

$$\begin{aligned} &\max_{x\in D} f(x,\alpha), \\ &\text{s.t. } g^j(x,\alpha) = c^j, \end{aligned}$$

for all $j = 1,\dots,m$. Assume $x(\alpha)$ and the Lagrange multipliers $\lambda^j(\alpha)$ are C^1 in α and the Jacobian of the constraints $\left(\frac{\partial g^j}{\partial x_i}(x^*(\alpha),\alpha)\right)$ has a rank equal to m.

Theorem 15. *Let $V(\alpha) = \max_{x\in D} f(x,\alpha)$ subject to $g^j(x,\alpha) = c^j$. Assume that $x(\alpha)$ and $V(\alpha)$ are differentiable in α_s. Then*

$$V_{\alpha_s}(\alpha) = f_{\alpha_s}(x(\alpha),\alpha) - \lambda^* g_\alpha(x(\alpha),\alpha).$$

Let $v(p,x)$ be the indirect utility function defined by $v(p,x) = \max_q u(q)$ subject to $\sum_i p_i q_i = x$ where p_i is the price of good i, x is the total income, q is the quantity demanded and $u(q)$ is a utility function. Then by the envelope theorem,

$$\begin{aligned} v_{p_i}(p,x) &= -\lambda q_i^*(p,x), \\ v_x(p,x) &= \lambda. \end{aligned}$$

Taking ratios,

$$q_i^*(p,x) = -\frac{v_{p_i}(p,x)}{v_x(p,x)},$$

which is better known as Roy's identity.

Proof. Let $V(\alpha) = f(x(\alpha), \alpha)$. Then

$$V_{\alpha_s}(\alpha) = \sum_i \frac{\partial f}{\partial x_i}(x^*(\alpha), \alpha)\frac{\partial x_i(\alpha)}{\partial \alpha_s} + f_{\alpha_s}(x(\alpha), \alpha).$$

Substituting the first order conditions into this differential gives,

$$\begin{aligned} V_{\alpha_s}(\alpha) &= \sum_i \sum_j \lambda^{j*} g^j{}_{x_i}(x(\alpha), \alpha)\frac{\partial x_i(\alpha)}{\partial \alpha_s} + f_{\alpha_s}(x(\alpha), \alpha), \\ &= \sum_j \lambda^{j*} \sum_i \left[g^j_{x_i}(x^*(\alpha), \alpha)\frac{\partial x_i(\alpha)}{\partial \alpha_s} \right] + f_{\alpha_s}(x(\alpha)). \end{aligned}$$

As $g_j(x^*(\alpha), \alpha) = c^j$ we have that,

$$\sum_i g^j_{x_i}(x(\alpha), \alpha)\frac{\partial x_i(\alpha)}{\partial \alpha_s} + g^j_{\alpha_s}(x(\alpha), \alpha) = 0.$$

As such,

$$V_{\alpha_s}(\alpha) = -\sum_j \lambda^*_j g^j_{\alpha_s}(x(\alpha), \alpha) + f_{\alpha_s}(x^*(\alpha),$$

$\square$

Exercises

Unconstrained Optimization Find the extreme points of the following functions.

- $f(x, y) = x^3 + y^3 - 3xy$. (sol: $(0, 0)$ saddle point and $(1, 1)$ a local minimum point)
- $f(x, y) = y^2 + xy\ln(x)$. (sol: $(1, 0)$ saddle point and $(1/e, 1/(2e))$ minimum point)
- $f(x, y, z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz$. (sol: $(0, 0, 0)$ minimum point)

Constrained optimization Optimize the following functions.

- $f(x, y) = xy$ subject to $4x + 2y = 80$ (sol: $(10, 20)$ maximum point).
- $f(x, y) = (x - 1)^2 + y^2 + z^2$ subject to $x + y + z = 4$. (sol: $(2, 1, 1)$ minimum point)
- $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x + y + z = 3$ and $z = 1$. (sol: $(1, 1, 1)$ minimum point)

Inequality Constraints Optimize the following functions.

- $f(x) = x^2 - 5x + 6$ subject to $x \geq 0$. (sol: $(5/2)$ a mininmum and 0 a (local) maximum).
- $f(x,y) = (x-1)^2 + (y-2)^2$ subject to $x \geq 0$ and $y \geq 0$. (sol: $(1,0)$ saddle point, (0,2) saddle point, (0,0) maximum point and $(1,2)$ minimum point.)
- $f(x,y) = 4 - x - y$ subject to $x^2 + y^2 \leq 1$. (sol: $(-\sqrt{1/2}, -\sqrt{1/2})$ maximum and $(\sqrt{1/2}, \sqrt{1/2})$ minimum).
- $f(x,y) = 2x + 2y - x^2 - y^2$ subject to $x + y \leq 1, x \geq 0, y \geq 0$. (sol: $(0,0)$ which is the lowest, $(0,1)$, $(1,0)$ and $(1/2, 1/2)$ which is the highest.)

Differential equations

Economists are often interested in studying the change in economic variables, e.g. national income, interest rate, employment, inflation. The law of motion that governs these variables is often expressed in terms of one or more equations.

If time is considered as continuous and the equations involve unknown functions and their derivatives, we are considering so called differential equations. Such equations are often used in macroeconomic theory but also pop up in many other areas of economics.

As the name suggest a differential equation is an equation. However, unlike usual algebraic equations, for differential equations the unknown is a function and the equation includes one or more of the derivatives of this 'unknown' function.

To begin, we restrict ourselves to first-order differential equations, that is equations where only the first-order derivative of the unknown functions of one variable are included.

$$y' = F(y, t).$$

Here y is considered to be a function of a single variable (say t) and we use the notation y' to denote the derivative $\frac{\partial y(t)}{\partial t}$. You can think about the variable t as time and $y(t)$ as the path of some variable y over time.

The following are some problems one might encounter.

1. Find all possible solutions, i.e. functions $y(t)$ that satisfy some differential equation.

2. **Cauchy problem:** find a (the) solution $y(t)$ that solves the differential equation and additionally meets a certain set of initial conditions, e.g. $y(t_0) = y_0, y'(t_0) = y'_0, \ldots$.

3. **Limits problem:** find a solution $y(t)$ of the differential equation, defined on the interval $[t_0, t_f]$ such that $y(t_0) = y_0$ and $y(t_f) = y_f$.

Typical examples are

$$y' = ay, \qquad y' + ay = b \qquad y' + ay = by^2.$$

For an economic example, assume the capital stock of a firm evolves according to the differential equation,

$$k'(t) = i(t) - \delta k(t),$$

where $i(t)$ is the rate of investment at time t and δ is the instantaneous depreciation rate. A natural question is: given some initial amount of captial $k(0) = k_0$ and a path of investments $i(t)$, what is the capital stock $k(t)$ at a certain point in time.

The general solution to a particular differential equation gives a formula for $y(t)$ that may contain some parameters whose particular value allows one to solve any Cauchy problem that may be encountered. For some types of equations, there are theorems demonstrating existence and uniqueness of a solution to each Cauchy problem. Many such theorems are "local" (assuming existence and uniqueness in a neighbourhood). Other theorems (strong hypotheses) are "global" or "regional". For some (privileged) types of differential equations (e.g. linear differential equations with constant coefficients and continuous right hand side) there is existence and uniqueness of a solution to any Cauchy problem, anywhere.

Although we often use t to denote the independent variable because most differential equations that appear in economics have time as their independent variable, it is interesting to realize that the theory is also valid even if the independent variable is not time. For example, in the literature on optimal nonlinear taxation, optimal tax schedules are often characterized by differential equations of some function $\tau(y)$ where y is pre-tax income and $\tau(y)$ gives the tax rate for income level y.

First order differential equations

A FIRST ORDER differential equation can be written as an equation of the following form,

$$P(y,t)y' + Q(y,t) = 0.$$

Here, $P(y,t)$ and $Q(y,t)$ are two functions of two variables, y and t and where and $y = y(t)$ is the unknown function of interest.

A PARTICULAR CLASS of first order differential equations are the **exact differential equations**. The differential equation $P(y,t)y' + Q(y,t) = 0$ is exact if there exist a function $\mu(y,t)$ of the two variables y and t such that,[31]

$$\mu_y(y,t) = P(y,t),$$
$$\mu_t(y,t) = Q(y,t).$$

An exact differential equation has the implicit solution,

$$\mu(y(t),t) = c,$$

Indeed, taking the derivative of both sides with respect to t gives,

$$\mu_y(y(t),t)y' + \mu_t(y(t),t) = 0,$$
$$\leftrightarrow P(y,t)y' + Q(y,t) = 0.$$

Young's theorem[32] tells us that when a differential equation is exact, then for all (y,t) it must be that,

$$P_t(y,t) = \mu_{y,t}(y,t) = \mu_{t,y}(y,t) = Q_y(y,t).$$

In fact, it turns out, that this is also a sufficient condition for the differential equation to be an exact equation.

[31] Here we use the notation,

$$\frac{\partial \mu(y,t)}{\partial y} = \mu_y(y,t),$$
$$\frac{\partial \mu(y,t)}{\partial t} = \mu_t(y,t).$$

[32] Young's theorem states that if $f(x,y)$ is C^2 then,

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}.$$

An exact differential equation can be solved using the following steps.

1. First, you check the condition $P_t(y,t) = Q_y(y,t)$ in order to make sure that the differential equation is indeed exact.

2. Next you integrate $\mu_t(y,t) = Q(y,t)$ with respect to t (over the interval $[t_0, t]$) to obtain an expression for $\mu(y,t)$, i.e.

$$\mu(y,t) - \mu(y,t_0) = \int_{t_0}^{t} Q(y,\tau)d\tau.$$

3. Next, we differentiate both sides of this identity with respect to y, giving[33]

$$\frac{\partial\mu(y,t)}{\partial y} - \frac{\partial\mu(y,t_0)}{\partial y} = \int_{t_0}^{t} \frac{\partial Q(y,\tau)}{\partial y}dt = \int_{t_0}^{t} \frac{\partial P(y,\tau)}{\partial \tau}dt = P(y,t) - P(y,t_0).$$

The last equality follows from the equality of $Q_y(y,t)$ and $P_t(y,t)$. Also $\frac{\partial\mu(y,t)}{\partial y} = P(y,t)$ so we obtain the equation,

$$\frac{\partial\mu(y,t_0)}{\partial y} = P(y,t_0).$$

4. Integrating both parts of the identity above, gives,

$$\mu(y,t_0) - \mu(y_0,t_0) = \int_{y_0}^{y} P(\gamma,t_0)d\gamma.$$

5. Finally, $\mu(y,t_0)$ can be substituted back into the condition obtained in step 2 to obtain,

$$\mu(y,t) = \int_{t_0}^{t} Q(y,\tau)d\tau + \int_{y_0}^{y} P(\gamma,t_0)d\gamma + \mu(y_0,t_0).$$

where $\mu(y_0,t_0)$ is a constant number.

From this, we see that the solution is given by,

$$\int_{t_0}^{t} Q(y,\tau)d\tau + \int_{y_0}^{y} P(\gamma,t_0)d\gamma + \mu(y_0,t_0) = C,$$

for some constant C. Plugging in $t = t_0, y(t_0) = y_0$ gives,

$$\mu(y_0,t_0) = C,$$

So we get the solution,

$$\int_{t_0}^{t} Q(y,\tau)d\tau + \int_{y_0}^{y} P(\gamma,t_0)d\gamma = 0.$$

[33] Observe the exchange of integration and differentiation.

Consider the exact differential equation

$$(2t^3 + 3y) + (3t + y - 1)y' = 0.$$

Here $Q(y,t) = 2t^3 + 3y$ and $P(y,t) = 3t + y - 1$. It is easily verified that this equation is exact. The solution gives,

$$\int_{t_0}^{t}(2\tau^3 + 3y)d\tau + \int_{y_0}^{y}(3t_0 + \gamma - 1)d\gamma = 0.$$

Integrating out gives,

$$\begin{aligned} &\frac{t^4}{2} + 3yt - \frac{t_0^4}{2} - 3yt_0 + 3t_0y, \\ &+ \frac{y^2}{2} - y - 3t_0y_0 + \frac{y_0^2}{2} + y_0 = 0 \\ \leftrightarrow &\frac{t^4}{2} + 3yt - \frac{t_0^4}{2} + \frac{y^2}{2} - y - 3t_0y_0 + \frac{y_0^2}{2} + y_0 = 0 \end{aligned}$$

The solution is therefore,

$$\frac{t^4}{2} + 3yt + \frac{y^2}{2} - y = \frac{t_0^4}{2} + 3t_0y_0 - \frac{y_0^2}{2} - y_0.$$

A DIFFERENTIAL EQUATION is called **separable** if it is of the form,

$$g(y)y' + f(t) = 0.$$

for two function g and f. Setting $P(y,t) = g(y)$ and $Q(y,t) = f(t)$ it is easily verified that these equations are exact differential equations.[34]

[34] Indeed, $P_t = 0 = Q_y$.

Using the procedure set out above, we get the following solution,

$$\begin{aligned}
&\int_{t_0}^{t} f(\tau)d\tau + \int_{y_0}^{y} g(\gamma)d\gamma = 0,\\
\leftrightarrow &F(t) - F(t_0) + G(y) - G(y_0) = 0,\\
\leftrightarrow &G(y) = G(y_0) + F(t) - F(t_0).
\end{aligned}$$

where F and G are the integrands of f and g.

In practice separable differential equations are also 'quickly' solved in the following way:

1. First separate out all functions that depend on y on one side and all functions that depend on t on the other side,

$$g(y)y' = -f(t),$$

2. Next we integrate both sides of the equation with respect to t,

$$\int_{t_0}^{t} g(y)y'(t)dt = \int_{t_0}^{t} f(t)dt.$$

3. By setting $y = y(t)$, we can use a change of variables, $\frac{dy}{dt} = y'(t)$ and $y(t_0) = y_0$, so:

$$\begin{aligned}
&\int_{y(t_0)}^{y(t)} g(y)dy = \int_{t_0}^{t} f(t)dt,\\
\Longleftrightarrow\ &G(y(t)) - G(y(t_0)) = F(t) - F(t_0).
\end{aligned}$$

Where G is a primitive of $g(y)$ and F is a primitive of f.

Separable differential equations are often 'quickly' solved as follows:

$$\begin{aligned}
&y' = \frac{t}{3y^2},\\
\rightarrow &3\int y^2 dy = \int t dt + C,\\
\rightarrow &y^3 = t^2/2 + C, \ldots
\end{aligned}$$

where C is a constant of integration. It is certainly faster, but can hide

Consider the separable differential equation $y' = \frac{t}{3y^2}$ with initial condition $y(t_0) = y_0$.

$$\begin{aligned}
&y' = \frac{t}{3y^2},\\
\rightarrow &3y^2y' = t,\\
\rightarrow &3\int_{t_0}^{t} y^2(t)y'(t)dt = \int_{t_0}^{t} tdt,\\
\rightarrow &3\int_{y(t_0)}^{y(t)} y^2 dy = \int_{t_0}^{t} tdt,\\
\rightarrow &y(t)^3 - y(t_0)^3 = \frac{t^2}{2} - \frac{t_0^2}{2},\\
\rightarrow &y(t)^3 = t^2/2 - t_0^2/2 + y(t_0)^3,\\
\rightarrow &y = \left(t^2/2 - t_0^2/2 + y_0^3\right)^{1/3}.
\end{aligned}$$

The Cauchy solution with initial values $t_0 = 0$ and $y(t_0) = 3$ gives,

$$y(t) = \left(t^3/2 + 27\right)^{1/3}.$$

problems of existence of the $\int$ involved. For example,

$$
\begin{aligned}
y' &= -\frac{y}{2t}, \\
\rightarrow \frac{1}{y} dy &= -\frac{1}{2t} dt, \\
\rightarrow \int \frac{1}{y} dy &= -\frac{1}{2} \int \frac{dt}{t} + C, \\
\rightarrow ?
\end{aligned}
$$

If $t_0 > 0$ and $y_0 > 0$. Then we have $\ln(y) - \ln(y_0) = -1/2(\ln(t) - \ln(t_0))$ which gives $y = y_0\sqrt{t_0/t}$. If $t_0 > 0$ and $y_0 < 0$, we have $\ln(-y) - \ln(-y_0) = -1/2(\ln(t) - \ln(t_0))$ so $y = y_0\sqrt{t_0/t}$. The other cases are treated similarly, however the solutions are in separate regions of the solution space. Notice that $y(t) = 0.$ is a solution if $y_0 = 0$.

For an economic inspired example, let $w(t) \in \mathbb{R}_{++}$ be the wealth held in an investment account at time t and let $r(t)$ the (instantaneous) interest rate with interest continuously compounded and initial value $w(t_0) = w_0$. Then,

$$w' = r(t)w.$$

This is a separable equation. Separating the variables and integrating gives,

$$
\begin{aligned}
\int_{t_0}^{t} \frac{w'(s)}{w(s)} ds &= \int_{t_0}^{t} r(s) ds, \\
\int_{w(t_0)}^{w(t)} \frac{1}{k} dk &= \int_{t_0}^{t} r(s) ds, \\
[\ln(k)]_{w(t_0)}^{w(t)} &= \int_{t_0}^{t} r(s) ds.
\end{aligned}
$$

As such, $\ln(w(t)) = R(t) + \ln(w(t_0))$ where $R(t) = \int_{t_0}^{t} r(s)ds$.

$$
\begin{aligned}
w(t) &= e^{R(t)+\ln(w_0)} = w_0 e^{R(t)}, \\
&= w_0 e^{\int_{t_0}^{t} r(s)ds}.
\end{aligned}
$$

A FUNCTION $f(y,t)$ is homogeneous of degree n if for all $\alpha > 0$, $f(\alpha y, \alpha t) = \alpha^n f(y,t)$. If for the differential equation

$$P(y,t)y' + Q(y,t)dt = 0,$$

both functions, P and Q are homogeneous of the same degree, we call it a **homogeneous differential equation**. This family of differential equations can be solved in the following way,

1. rewrite the equation as follows,

$$
\begin{aligned}
&P(y,t)y' + Q(y,t) = 0, \\
\rightarrow &t^n P(y/t,1)y' + t^n Q(y/t,1) = 0, \\
\rightarrow &P(y/t,1)y' + Q(y/t,1) = 0.
\end{aligned}
$$

2. Introduce the change of variables $u(t) = y(t)/t$, then,

$$
\begin{aligned}
y &= ut, \\
\rightarrow y' &= u + u't.
\end{aligned}
$$

   Substituting this in the differential equation gives,

$$
\begin{aligned}
&P(u,1)(u+u't) + Q(u,1) = 0, \\
\rightarrow &P(u,1)u't = -[Q(u,1) + P(u,1)u], \\
\rightarrow &-\frac{P(u,1)}{Q(u,1)+P(u,1)u} u' - \frac{1}{t} = 0.
\end{aligned}
$$

3. This is a separable equation that can be solved in the usual way. Finally, we substitute $u(t) = y(t)/t$ into the solution for u in order to obtain a solution for $y(t)$.

As an example, let

$$
\begin{aligned}
y' &= -\frac{t^2+y^2}{2ty}, \\
\rightarrow y' &= -\frac{1+(y/t)^2}{2(y/t)}.
\end{aligned}
$$

Let $u = y/t$, then,

$$
\begin{aligned}
u't + u &= -\frac{1+u^2}{2u}, \\
\iff u' &= -\frac{1}{t}\frac{1+3u^2}{2u}, \\
\rightarrow \int_{u_0}^{u} \frac{2u}{1+3u^2} du &= -\int_{t_0}^{t} \frac{1}{t} dt.
\end{aligned}
$$

For the left hand side, consider a change of variables $v = 1 + 3u^2$ then $dv = 6udu$ so

$$\int_{1+3u_0^2}^{1+3u^2} \frac{1}{3v} dv = -\int_{t_0}^{t} \frac{1}{t} dt.$$

Then

$$
\begin{aligned}
\frac{1}{3}\ln\left(\frac{1+3u^2}{1+3u_0^2}\right) &= -\ln(t) + \ln(t_0), \\
\rightarrow u^2 &= \frac{1}{3}\left[(t_0/t)^3(1+3u_0^2) - 1\right], \\
\rightarrow y^2 &= \frac{t^2}{3}\left[(t_0/t)^3(1+3u_0^2) - 1\right].
\end{aligned}
$$

IN MANY CASES, the differential equation $P(y,t)dy + Q(y,t)dt = 0$ is neither exact nor homogeneous. In such settings, it might still be possible to find what we call an **integrating factor**. An integrating factor is a function $z(y,t)$ of y and t such that the differential equation

$$z(y,t)P(y,t)dy + z(y,t)Q(y,t)dt = z(y,t)[P(y,t)dy + Q(y,t)dt] = 0,$$

becomes an exact equation.

Young's theorem tells us that z is an integrating factor iff

$$\begin{aligned} &\frac{d}{dt}zP = \frac{d}{dy}zQ, \\ \leftrightarrow &z_t P + zP_t = z_y Q + zQ_y, \\ \leftrightarrow &z_t P - z_y Q + z(P_t - Q_y) = 0. \end{aligned}$$

In general this is not an easy problem to solve (given that it is also a differential equation). However, there are two special cases of particular interest. A first case is when z is independent of y. Then, $z_y = 0$ so,

$$\begin{aligned} &z_t P = z(Q_y - P_t), \\ &\frac{1}{z}z_t = \frac{Q_y - P_t}{P}. \end{aligned}$$

The left hand side is independent of y so this can only be a solution is the right hand side is also independent of y, i.e. there should be a function G such that,

$$\frac{Q_y - P_t}{P} = G(t).$$

In this case, we have that $\frac{1}{z}z_t = G(t)$, so integrating both sides gives,[35]

$$\begin{aligned} &\ln(z) = \int_{t_0}^{t} G(s)ds, \\ \rightarrow &z(t) = e^{\int_{t_0}^{t} G(s)ds}. \end{aligned}$$

[35] Here we omit the integrating constant $\ln(z_0)$ as this does not change the fact that $z(t)$ remains an integrating factor.

The second special case is when z is independent of t. Then, $z_t = 0$ so

$$\begin{aligned} &z_y Q = z(P_t - Q_y), \\ &\frac{z_y}{z} = \frac{P_t - Q_y}{Q} = H(y). \end{aligned}$$

Integrating out gives

$$\begin{aligned} &\ln(z) = \int_{y_0}^{y} H(s)ds, \\ \rightarrow\ &z(y) = e^{\int_{y_0}^{y} H(s)ds} \end{aligned}$$

The equation

$$(t^2 + y^2 + t) + tyy' = 0,$$

is not exact. However,

$$\frac{Q_y - P_t}{P} = 1/t = G(t),$$

so $z(t) = e^{\int 1/t dt}$ is an integrating factor. However, $e^{\int \frac{1}{t} dt}$ is equal to t for $t > 0$ and equal to $-t$ if $t < 0$. So,

$$\begin{aligned} &t[(t^2 + y^2 + t) + tyy'] = 0, \\ &(t^3 + y^2 t + t^2) + t^2 yy' = 0. \end{aligned}$$

This equation is exact. We have

$$\begin{aligned} &\int_{t_0}^{t} (\tau^3 + y^2\tau + \tau^2)d\tau + \int_{y_0}^{y} (t_0^2\gamma)d\gamma = 0, \\ \leftrightarrow &\frac{t^4}{4} + \frac{y^2t^2}{2} + \frac{t^3}{3} = \frac{t_0^4}{4} + \frac{t_0^3}{3} + t_0^2\frac{y_0^2}{2} \end{aligned}$$

A FIRST ORDER differential equation is **linear** if it is of the form,

$$y' + u(t)y = w(t),$$

where $u(t)$ and $w(t)$ may be functions of t.[36] Let us first focus on the simplest case where $u(t)$ and $w(t)$ are constants,

$$y' + uy = w.$$

Here $P(y,t) = 1$ and $Q(y,t) = (uy - w)$. Then $(Q_y - P_t)/P = u$ so we have that $z(t) = e^{ut}$ is an integrating factor. Multiplying the differential equation by e^{ut} gives,

$$y'e^{ut} + uye^{ut} - we^{ut} = 0.$$

The solution is then given by,

$$\begin{aligned}
&\int_{t_0}^{t}(uy - w)e^{u\tau}d\tau + \int_{y_0}^{y} e^{ut_0}d\gamma = 0,\\
\leftrightarrow &(y - w/u)e^{ut} - (y - w/u)e^{ut_0} + (y - y_0)e^{ut_0} = 0,\\
\leftrightarrow &ye^{ut} = \frac{w}{u}e^{ut} - \frac{w}{u}e^{ut_0} + y_0e^{ut_0},\\
\leftrightarrow &y = \frac{w}{u} + (y_0 - \frac{w}{u})e^{-u(t-t_0)}
\end{aligned}$$

If $u \geq 0$ and t becomes very big, then y will be close to w/u.

Another way to solve the problem is by starting again from the equation,

$$y'e^{ut} + uye^{ut} - we^{ut} = 0.$$

Now write $y = y(t)$ and notice that the left hand side is the (total) derivative of $(e^{ut}y(t))$ with respect to t, so

$$\frac{d}{dt}\left[e^{ut}y(t)\right] = we^{ut}.$$

Then integrating both sides with respect to t gives,

$$e^{ut}y(t) - e^{ut_0}y(t_0) = \int_{t_0}^{t} we^{ut}dt = \frac{w}{u}e^{ut} - \frac{w}{u}e^{ut_0}.$$

where C is a constant of integration. As such,

$$y(t) = \frac{w}{u} + (y(t_0) - \frac{w}{u})e^{-u(t-t_0)},$$

Let us now allow the right hand side $w(t)$ to depend on t. We can use the same factor $z(t) = e^{ut}$ as an integrating factor to obtain the

[36] Here are some examples of first order linear equations:

$$\begin{aligned}
&y' + y = t,\\
&y' + 2ty = 4t,\\
&(t^2 + 1)y' + e^t y = t\ln t.
\end{aligned}$$

Take the equation

$$y' + 2y = 8.$$

The integrating factor is e^{2t}. Then the solution is given by

$$y(t) = 4 + (y_0 - 4)e^{-2(t-t_0)}.$$

For a more economic inspired example, consider a capital stock where $k' = 2 - 0.1k$. The integrating factor is $e^{0.1t}$. The solution is,

$$k(t) = 20 + (k_0 - 20)e^{-0.1(t-t_0)}.$$

solution

$$\begin{aligned}
&\int_{t_0}^{t} (uy - w(\tau))e^{u\tau} d\tau + \int_{y_0}^{y} e^{ut_0} d\gamma = 0, \\
\leftrightarrow & y(e^{ut} - e^{ut_0}) - \int_{t_0}^{t} w(\tau)e^{u\tau} d\tau + (y - y_0)e^{ut_0} = 0, \\
\leftrightarrow & ye^{ut} = \int_{t_0}^{t} w(\tau)e^{u\tau} d\tau + y_0 e^{ut_0}, \\
\leftrightarrow & y = \int_{t_0}^{t} w(\tau)e^{-u(t-\tau)} d\tau + y_0 e^{-u(t-t_0)}.
\end{aligned}$$

Let us now consider the general case where both $u(t)$ and $w(t)$ are functions of t. The integrating factor is now equal to,

$$z(\tau) = e^{\int_{t_0}^{\tau} u(s)ds},$$

Then

$$\begin{aligned}
&\int_{t_0}^{t} z(\tau)u(\tau)y d\tau - \int_{t_0}^{t} z(\tau)w(\tau)d\tau + \int_{y_0}^{y} z(t_0)d\gamma = 0, \\
\leftrightarrow & y\int_{t_0}^{t} z(\tau)u(\tau)d\tau - \int_{t_0}^{t} z(\tau)w(\tau)d\tau + (y - y_0) = 0.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{t_0}^{t} z(\tau)u(\tau)d\tau &= \int_{t_0}^{t} e^{\int_{t_0}^{\tau} u(s)ds} u(\tau)d\tau, \\
&= z(t) - z(t_0) = z(t) - 1.
\end{aligned}$$

As such,

$$\begin{aligned}
& yz(t) = \int_{t_0}^{t} z(\tau)w(\tau)d\tau + y_0 = 0, \\
\leftrightarrow & y = e^{-\int_{t_0}^{t} u(s)ds} \int_{t_0}^{t} e^{\int_{t_0}^{\tau} u(s)ds} w(\tau)d\tau + e^{\int_{t_0}^{t} u(s)ds} y_0, \\
\leftrightarrow & y = \int_{t_0}^{t} e^{-\int_{\tau}^{t} u(s)ds} w(\tau)d\tau + e^{\int_{t_0}^{t} u(s)ds} y_0,
\end{aligned}$$

Assume that the amount of saving in an account at time t is gvien by $w(t)$. Suppose there are deposits and withdrawals at the rates $y(t)$ and $c(t)$. If there is continuous compounding of interest at rate $r(t)$ then the wealth at time t follows a differential equation,

$$w'(t) = r(t)w(t) + y(t) - c(t).$$

This is a first order linear differential equation. The solution is given by,

$$\begin{aligned}
w(t) &= w_0 e^{\int_0^t r(s)ds}, \\
&+ e^{\int_0^t r(s)ds} \int_0^t [y(k) - c(k)] e^{\int_0^k -r(s)ds} dk, \\
&= w_0 e^{\int_0^t r(s)ds} + \int_0^t [y(k) - c(k)] e^{\int_k^t r(s)ds} dk,
\end{aligned}$$

The first term gives the added value to $w(t)$ of the initial endowment $w(0)$. For the second term, notice that at point k, an amount $y(k) - c(k)$ is added to the savings. In time t the worth of this savings is equal to $(y(k) - c(k))e^{\int_k^t r(s)ds}$.

As a final case of a first order differential equation, we consider **Bernoulli equations**. These have the form,

$$y' + u(t)y = w(t)y^m.$$

If $m = 0$ this reduces to a linear equation. If $m = 1$ this equation is separable. Observe that $y(t) = 0$ is a solution, so let $y \neq 0$ and divide both sides by y^m to get,

$$\frac{y'}{y^m} + u(t)y^{1-m} = w(t).$$

Now, we introduce a change of variables. Let $z = y^{1-m}$. Then $z' = (1-m)\frac{y'}{y^m}$ so,

$$\frac{z'}{1-m} + u(t)z = w(t),$$
$$z' + (1-m)u(t)z = (1-m)w(t).$$

This is a linear differential equation.

Consider the differential equation

$$y' + ty = 3ty^2.$$

Observe that $y(t) = 0$ is a solution. If $y(t) \neq 0$, divide by y^2 and let $z = y^{-1}$ then $z' = -y^{-2}y'$. So,

$$-z' + tz = 3t,$$
$$z' - tz = -3t.$$

The latter equation is linear but also separable.

$$\frac{z'}{z-3} = t.$$

The solution is $\ln(z-3) - \ln(z_0 - 3) = t^2/2 - t_0^2/2$ (provided $z > 3$). If $z < 3$ we have the solution $-\ln(3-z) + \ln(3-z_0) = t^2/2 - t_0^2/2$. In both cases,

$$z - 3 = (z_0 - 3)(e^{t^2/2 - t_0^2/2}.$$

Then

$$y = \frac{1}{3 + (z_0 - 3)e^{t^2/2 - t_0^2/2}}.$$

FOR A MORE ELABORATE example, consider the capital stock k (with initial stock k_0) that produces an output y according to the production function $y = k^\alpha$. A fraction s of the output is reinvested and capital depreciates at a constant rate equal to δ. This gives the dynamic equation,

$$k' = sk^\alpha - \delta k,$$
$$\leftrightarrow k'k^{-\alpha} + \delta k^{1-\alpha} = s.$$

Introduce the variable $u = k^{1-\alpha}$ with $u' = (1-\alpha)k^{-\alpha}k'$. Substituting gives,

$$\frac{u'}{1-\alpha} + \delta u = s,$$
$$\leftrightarrow u' + (1-\alpha)\delta u = s(1-\alpha).$$

This is a linear differential equation with solution,

$$u = \frac{s}{\delta} + (u_0 - s/\delta)e^{-(1-\alpha)\delta(t-t_0)}.$$

Setting $u_0 = k_0^{1-\alpha}$, $u = k^{1-\alpha}$ and $t_0 = 0$ gives,

$$k^{1-\alpha} = \frac{s}{\delta} + (k_0^{1-\alpha} - s/\delta)e^{-(1-\alpha)\delta t)}.$$
$$\rightarrow k(t) = \left(k_0^{1-\alpha}e^{-(1-\alpha)\delta t} + \frac{s}{\delta}\left(1 - e^{-(1-\alpha)\delta t}\right)\right)^{1/(1-\alpha)}$$

Also,

$$y(t) = \left(y_0^{(1-\alpha)/\alpha}e^{-(1-\alpha)\delta t} + \frac{s}{\delta}\left(1 - e^{-(1-\alpha)\delta t}\right)\right)^{\alpha/(1-\alpha)}.$$

Captial (and output) is a weighted average of initial and steady state and initial values. The speed of convergence is governed by the factor $(1-\alpha)\delta$.

Qualitative theory and stability

It is of course very convenient when economic models have differential equations that can be solved explicitly. Unfortunately, most types of differential equations do not have this nice property. If so, the nature of their solution has to be investigated in some other way.

Many equations in economic models have the following form,

$$y' = F(y).$$

This is called an autonomous equation as t does not explicitly appear in the function F. In other words, the change in y, i.e. y' depends on the level of y but not on the exact time. To examine the properties of the solution to this equation, it is useful to study its phase diagram. This is obtained by drawing the curve $F(y)$. For all values of y for which $F(y) > 0$, the value of y will be increasing (as $y' > 0$) for all values of y for which $F(y) < 0$, the value of y will be decreasing (as $y' < 0$).

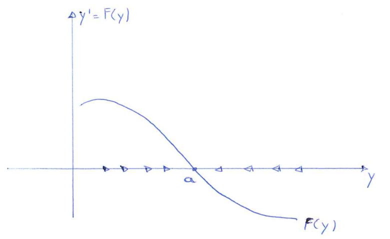

Figure 6: A dynamic model with one stationary state.

At a point $y = a$ where $F(a) = 0$, the value of y will not change. Such point is called a stationary state. If the system is in such stationary state, it will not move from this point.

Figure 7 shows a setting with two stationary states (a and b). These stationary states are quite different. In particular, if we start very close to a then $y(t)$ will approach a as time goes by. On the other hand, if x starts close to b, it will move further away from b as time goes by. We call the stationary state a a stable state and b an unstable state.

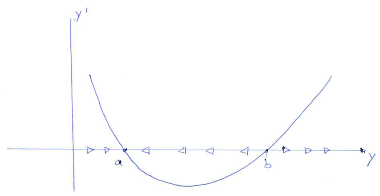

Figure 7: A dynamic model with two stationary state.

Example: Let $y' + ay = b$. Then this is an autonomous system with $F(y) = b - ay$. There is one stationary state when $y = b/a$. Also $F'(b/a) = -a$ so this stationary state is stable if $a > 0$ and is unstable if $a < 0$.

So, if $F(a) = 0$ and $F'(a) < 0$ then a is a locally asymptotically stable state. If $F(a) = 0$ and $F'(a) > 0$ then a is a locally asymptotically unstable state. What happens if $F(a) = 0$ and $F'(a) = 0$. Then both can happen (there are four separate cases here, can you draw them?).

As an illustration we take the Solow-Swan model, which is a simple necoclassical growth model that involves a constant returns to scale production function $Y = F(K, L)$ that determines national output Y as a function of capital K and labour L (i.e. $F(tK, tL) = tF(K, L)$. It is assumed that L grows at a constant proportional rate $\lambda > 0$ (i.e. $L' = \lambda L$). Also, a constant fraction of output Y is devoted to net investment $K' = sY$. The model usually divides all variables by L so $y = Y/L$, $k = K/L$ giving,

$$\begin{aligned} Y &= F(K, L) = LF(K/L, 1), \\ \rightarrow y &= f(k). \end{aligned}$$

Also,

$$\begin{aligned} \frac{k'}{k} &= \frac{K'}{K} - \frac{L'}{L} = s\frac{Y}{K} - \lambda = s\frac{y}{k} - \lambda, \\ \rightarrow k' &= sf(k) - \lambda k. \end{aligned}$$

Without specifiying f, this equation has no explicit solution. Let us assume that $f(0) = 0$, $f'(k) > 0$ and $f''(k) < 0$. Provided that $sf'(0) > \lambda$ and $sf'(k) < \lambda$ for large k we can plot the phase diagram as a hump shaped function. There is a unique equilibrium state with $k^* > 0$ determined by $sf(k^*) = \lambda k^*$. Moreover, the stationary state will be stable.

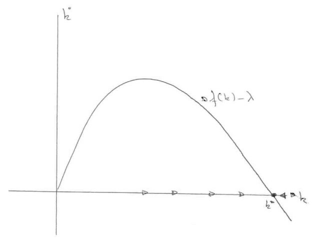

Figure 8: Dynamics in the Solow-Swan model.

Second order differential equations

ABOVE, WE ONLY considered first order differential equations. However, many economic models are based on differential equations in which second or higher order derivatives appear.

A typical second order differential equation takes the form

$$y'' = F(t, y, y').$$

Where $y'' = \frac{\partial^2 y(t)}{\partial t^2}$. A solution of this differential equation is a twice differentiable function that satisfies the equation.

Let $y'' = k$ with k a constant. Then integrating both sides gives $y' = kt + C$ where C is a constant of integration. Again integrating both sides gives $y(t) = \frac{k}{2}t^2 + Ct + D$.

A special case which is worth mentioning is when F does not depend on y. In this case we have $y'' = F(t, y')$. Introducing a new variable $u = y'$ gives $u' = F(t, u)$ which is a first order differential equation. As such, we will focus on settings where F depends on y. In particular, we will mainly focus on **second order linear differential equations** with constant coefficients,

$$y'' + ay' + y = f(t).$$

where $f(t)$ is a continuous functions of t and $a, b \in \mathbb{R}$.

In contrast to first order linear differential equations, there is no explicit general solution to this problem. However, something useful can be said about the structure of the general solution. Let us start by looking at the **homogeneous** equation,[37]

[37] This is the equation obtained by setting $f(t) = 0$.

$$y'' + ay' + by = 0,$$

that is obtained by replacing $f(t)$ by zero. Now, how does solving this homogeneous differential equation help us in solving the non-homogeneous equation $y'' + ay' + by = f(t)$ for $f(t) \neq 0$.

Well, assume that we know (or find) one particular solution $y^*(t)$ of the non-homogeneous equation, i.e. $y^*(t)$ satisfies

$$y''^* + ay'^* + by^* = f(t).$$

Then for any function $y(t)$ that also solves the differential equation, we see that the difference $y(t) - y^*(t) = u(t)$ solves the homogeneous differential equation, i.e.,

$$u'' + au' + bu = 0.$$

As such, any solution $y(t)$ of the non-homogeneous differential equation can be written as a general solution of the homogeneous equation $u(t)$ and a particular solution of the original differential equation $y^*(t)$.

$$y(t) = y^*(t) + u(t).$$

Given this, we can split up the problem of solving the second order linear differential equation in two parts.

1. Find a general solution of the homogeneous equation.
2. Find a particular solution of the differential equation.
3. Write the general solution of the differential equation as the sum of the particular solution and the general solution of the homogeneous equation.

In order to find the general solution to the homogeneous differential equation, we first make a short digression concerning the dimension of the solution space of this problem.

A first thing to notice about the solutions of the second order homogeneous linear differential equation is that it forms a vector space.[38] As such, for any two solutions y_1, y_2 of the homogeneous differential equation, the function $\alpha y_1 + \beta y_2$ also solve the differential equation.

[38] A vector space is a set which is closed by the operations of addition and scalar multiplication. In particular, assume that y solves the differential equation.

$$y'' + ay' + by = 0,$$

Then for all scalars α, we have that the function αy also solves the differential equation and if y_1 and y_2 both solve the differential equation, then $y_1 + y_2$ is also a solution.

Lemma 1. *Assume that the homogeneous differential equation*

$$y'' + ay' + by = 0,$$

has a solution on $[t_0, t_1]$ *with* $y(t_0) = y'(t_0) = 0$*, and* $a, b \in \mathbb{R}$*. Then* $y : [t_0, t_1] \to \mathbb{R}$ *is the constant function* $y(t) = 0$ *for all* $t \in [t_0, t_1]$*.*

Proof. Consider the function $\sigma : [t_0, t_1] \to \mathbb{R}$ with,

$$\sigma(t) = (y'(t))^2 + (y(t))^2.$$

Observe that $\sigma(t) \geq 0$ for all t and that $\sigma(t) = 0$ if and only if $y(t) = y'(t) = 0$. Additionally, $\sigma(t_0) = 0$ by assumption. Also σ is C^1 and taking derivatives gives,

$$\begin{aligned}
\sigma' &= 2y'y'' + 2yy' = 2y'(-ay' - by) + 2yy', \\
&= -2a(y')^2 + 2(1-b)yy', \\
&\leq |2a|(y')^2 + 2|(1-b)||yy'|,
\end{aligned}$$

Now, for any two numbers z, w,

$$0 \le (|z| - |w|)^2 = z^2 + w^2 - 2|z||w|,$$
$$\leftrightarrow 2|z||w| \le z^2 + w^2.$$

As such,

$$\sigma' \le |2a|(y')^2 + |(1-b)|(y^2 + y'^2) \le A(y'^2 + y^2) = A\sigma.$$

where $A = |2a| + |1-b|$. Next, observe that,

$$\frac{d}{dt}[e^{-At}\sigma(t)] = e^{-At}\sigma'(t) - Ae^{-At}\sigma(t) = e^{-At}(\sigma'(t) - A\sigma(t)) \le 0.$$

for all $t \in [t_0, t_1]$. Now, given that $\sigma(t_0) = 0$ and the derivative is nowhere strictly, positive, we can integrate the left hand side shows that $e^{-At}\sigma(t) \le 0$.

This means that $\sigma(t) \le 0$ for all $t \in [t_0, t_1]$. Together with the fact that $\sigma(t) \ge 0$ for all t, we obtain that $\sigma(t) = 0$ for all $t \in [t_0, t_1]$. From this, it follows that $y'(t) = y(t) = 0$ for all $t \in [t_0, t_1]$, so y is the constant function equal to zero. □

The following result shows that a solution for the homogeneous equation only depends on the initial conditions.

Lemma 2. *For every values α, β, there is at most one solution y to the differential equation,*

$$y'' + ay' + by = 0,$$

with $y(t_0) = \alpha$ and $y'(t_0) = \beta$.

Proof. Assume that there are two solutions y_1, y_2 with $y_1(t_0) = y_2(t_0) = \alpha$ and $y_1'(t_0) = y_2'(t_0) = \beta$. Then $g(t) = y_1(t) - y_2(t)$ is also a solution to the differential equation and it satisfies the initial conditions $g(t_0) = 0 = g'(t_0)$. By the previous lemma, g is the zero function on $[t_0, t_1]$. As such, we have that for all $t \in [t_0, t_1]$,

$$0 = g(t) = y_1(t) - y_2(t),$$

which shows that y_1 coincides with y_2 on the entire interval. □

The important result shows that any solution to the homogeneous differential equation is the linear combination of two non-zero, non linear independent solutions.[39]

[39] In other words, the solution space to the homogeneous differential equation is of dimension two.

Theorem 16. *Let $y_1, y_2 : [t_0, t_1] \to \mathbb{R}$ be two linearly independent non-zero solutions to the homogeneous second order differential equation,*

$$y'' + ay' + by = 0,$$

i.e. there is no $\alpha \in \mathbb{R}$ such that $y_1 = \alpha y_2$. Then all solutions to the linear homogeneous equation can be written as $\alpha y_1 + \beta y_2$ for some $\alpha, \beta \in \mathbb{R}$.

Proof. Let y be a solution to the homogeneous differential equation. Let us first show that there are α, β such that $y(t_0) = \alpha y_1(t_0) + \beta y_2(t_0)$ and $y'(t_0) = \alpha y_1'(t_0) + \beta y_2'(t_0)$.

This is the case whenever,

$$\alpha = \left.\frac{yy_2' - y'y_2}{y_1y_2' - y_2y_1'}\right|_{t_0},$$
$$\beta = \left.\frac{y'y_1 - yy_1'}{y_1y_2' - y_2y_1'}\right|_{t_0}.$$

These solutions exist if and only if the denominator $y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)$ is not zero.[40] Given that the solutions are not zero, i.e. not both $y_1, y_1' = 0$ or $y_2, y_2' = 0$ we have that the denominator is zero if and only if $y_1(a) = ay_2(a)$ and $y_1'(a) = ay_2'(a)$ for some fixed number a. But then, the function $g = y_1 - ay_2$ has $g(t_0) = g'(t_0) = 0$ and therefore g is the zero function. This means that $y_1(t) = ay_2(t)$ for all $t \in [t_0, t_1]$ contradicting the assumption that the solutions were not proportional.

So we know that values α, β exist. Then the two solutions y and $g = \alpha y_1 + \beta y_2$ satisfy the initial conditions $y(t_0) = g(t_0)$ and $y'(t_0) = g'(t_0)$ so they must be (by the previous lemma) identical. □

[40] This is the determinant of the matrix

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$$

It is also known as the Wronskian.

ABOVE, WE SAW that in order to find the general solution of a homogeneous second order linear differential equation, we need to find two linearly independent and non-zero solutions to the homogeneous differential equation. For linear differential equations of order 1, we found that functions of the form e^{rt} worked quite well as integrating factor. So let us try a function of the form $y(t) = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substituting into the linear differential equation gives,

$$r^2e^{rt} + are^{rt} + be^{rt} = 0,$$
$$\rightarrow e^{rt}[r^2 + ar + b] = 0,$$
$$\rightarrow r^2 + ar + b = 0.$$

This is quadratic equation is called **the characteristic equation** of the homogeneous differential equation. The roots of this quadratic equation are given by,[41]

$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \qquad r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

These roots are real if and only if the discriminant $a^2 - 4b \geq 0$, if $a^2 - 4b = 0$ the two roots are equal and if $a^2 - 4b < 0$ the roots are complex numbers. Let us take each case in turn.

- If $a^2 - 4b > 0$ the two roots are real and distinct, say r_1 and r_2. This gives two linearly independent solutions $y_1(t) = e^{r_1t}$ and

[41] The roots are the points for which the equation is equal to zero. For a quadratic equation of the form $a_1x^2 + a_2x + a_3$, there are two (possibly complex) roots given by,

$$x_1 = \frac{-a_2 + \sqrt{a_2^2 - 4a_1a_3}}{2},$$
$$x_2 = \frac{-a_2 - \sqrt{a_2^2 - 4a_1a_3}}{2}.$$

The numbers $a_2^2 - 4a_1a_3$ is called the discriminant.

$y_2(t) = e^{r_2}t$. Given that any solution is a linear combination of two linearly independent solution, it follows that the general solution of the homogeneous differential equation can be written as,

$$y(t) = Ae^{r_1 t} + Be^{r_2 t},$$

where $A, B \in \mathbb{R}$.

- If $a^2 - 4b = 0$ then $r = -a/2$ is a double root and $y_1(t) = e^{rt}$ is one solution of the differential equation. We still need to find a second solution which is linear independent of this one. We claim that $y_2(t) = te^{rt}$ is also a solution. We have that $y_2' = e^{rt} + rte^{rt}$ and $y_2'' = re^{rt} + re^{rt} + r^2te^{rt}$ so,

$$\begin{aligned} 2re^{rt} + r^2te^{rt} + ae^{rt} + arte^{rt} + bte^{rt} &= e^{rt}\left[2r + r^2t + a + art + bt\right], \\ &= e^{rt}\left[(2r+a) + (r^2 + ar + b)t\right] = 0. \end{aligned}$$

the first is zero as $r = -a/2$ the second term is zero as r solves the characteristic function. This shows that the general solution of the homogeneous differential equation can be written as a linear combination of e^{rt} and te^{rt},

$$y(t) = Ae^{rt} + Bte^{rt}.$$

- If $a^2 - 4b < 0$ the two roots are complex numbers. Let $r = -a/2$ be the real part of the roots and let $\theta = \frac{\sqrt{|a^2-4b|}}{2}$ be the imaginary part. Then the roots are given by $r_1 = r + i\theta$ and $r_2 = r - i\theta$ where $i^2 = -1$ is the imaginary number. The solutions to the differential equations are given by,

$$y_1(t) = e^{rt+i\theta t} = e^{rt}e^{i\theta} \qquad y_2(t) = e^{rt-i\theta t} = e^{rt}e^{-i\theta}.$$

Now, $e^{i\theta t} = \cos(\theta t) + i\sin(\theta t)$ and $e^{-i\theta t} = \cos(\theta t) - i\sin(\theta t)$.[42] As such, we get the 'complex valued' solutions,

$$\begin{aligned} y_1(t) &= e^{rt}(\cos(\theta t) + i\sin(\theta t)), \\ y_2(t) &= e^{rt}(\cos(\theta t) - i\sin(\theta t)), \end{aligned}$$

which will both solve the homogeneous equation and they are linearly independent. However, we would like our solutions to be real valued. Given that any (complex) linear combination of the two solutions will also solve the equation, we can consider the linear combinations $(y_1(t) + y_2(t))/2 = e^{rt}\cos(\theta t)$ and $(y_1(t) - y_2(t))/(2i) = e^{rt}\sin(\theta t)$.[43] These two equations are also linearly independent and they solve the differential equation. As such, a general solution is given by,

$$y(t) = e^{rt}(A\cos(\theta t) + B\sin(\theta t)).$$

[42] The complex number $e^i\theta$ is the number on the unit circle with angle θ. As such, its coordinates are $\cos\theta$ and $\sin\theta$.

[43] We have $i^2 = -1$, so $1/i = -i$.

This covers all cases, so we can find a general solution for all second order homogeneous linear differential equations.

Let's solve $y'' - 3y = 0$. The characteristic equation is $r^2 - 3 = 0$ which has roots $r_1 = -\sqrt{3}$ and $r_2 = \sqrt{3}$. The general solution is

$$y(t) = Ae^{-\sqrt{3}t} + Be^{\sqrt{3}t}.$$

As a second example, let us $y'' - 4y' + 4y = 0$. The characteristic equation is $r^2 - 4r + 4$ which has roots $r_1 = r_2 = 2$. As such the solution is,

$$y(t) = Ae^{2t} + Bte^{2t}.$$

For an example with complex roots, consider $y'' - 6y' + 13y = 0$. The characteristic equation is $r^2 - 6r + 13$ which has roots $r_1 = 3 + i2$ and $r_2 = 3 - i2$. The solution is,

$$y(t) = e^{3t}(A \sin 2t + B \cos 2t).$$

Nonhomogeneous equations

NOW, CONSIDER THE non-homogeneous equation with constant coefficients,

$$y'' + ay' + by = f(t).$$

where $f(t)$ is an arbitrary function. The general solution is given by,

$$y(t) = Ay_1(t) + By_2(t) + y^*(t).$$

where $y_1(t), y_2(t)$ solve the homogeneous equation and $y^*(t)$ is a particular solution to the nonhomogeneous equation. How do we find a solution $y^*(t)$.

- If $f(t) = d$ is a constant, we check wheter there is a particular solution $y^*(t) = c$, also a constant. Indeed, this gives

  $$bc = d,$$

  or $c = d/b$. As such, a general solution is $y(t) = Ay_1(t) + By_2(t) + d/b$.

- If $f(t)$ is a polynomial of degree n, then a reasonable guess is that $y^*(t)$ can also be a polynomial of degree n, $y^*(t) = A_n t^n + A_{n-1}t^{n-1} + \ldots + A_0$. The undertermined coefficients $A_n, A_{n-1}, \ldots, A_0$ are determined by the differential equation by equating the powers of t.

- If $f(t)$ is of the form $De^{\alpha t}$ for some constant D, it is good to try a function of the form $y^*(t) = Ee^{\alpha t}$.

In general is a good idea to match the functional form of $f(t)$ as close as possible. There exist other methods to find particular solutions, namely the method of undetermined coefficients. An explanation of this method can be found in any good handbook on differential equations.

Let's solve $y'' - 4y' + 4y = t^2 + 2$. The right hand side is a polynomial of degree 2, so set $y^*(t) = At^2 + Bt + C$. We have $y'^* = 2At + B$ and $y''^* = 2A$, so

$$2A - 4(2At + B) + 4(At^2 + Bt + C) = t^2 + 2,$$

$$4At^2 + (8A - 4B)t + (2A - 4B + 4C) = t^2 + 2.$$

As such, equating powers gives $4A = 1$ or $A = 1/4$, $8A - 4B = 2 - 4B = 0$ or $B = 1/2$ and $(2A - 4B + 4C) = 1/2 - 2 + 4C = 2$ giving $C = 7/8$. So,

$$y^*(t) = t^2/4 + t/2 + 7/8.$$

FOR A SECOND order nonhomogeneous differential equation

$$y'' + ay' + by = f(t),$$

we obtain the general solution $y(t) = Ay_1(t) + By_2(t) + y^*(t)$. where $Ay_1(t) + By_2(t)$ is a general solution of the homogeneous equation and $y^*(t)$ is a particular solution of the non-homogeneous equation.

The differential equation is **globally asymptotically stable** if every solution $Ay_1(t) + By_2(t)$ of the homogeneous equation tends to zero as $t \to \infty$.[44] A necessary and sufficient condition for this is that $y_1(t) \to 0$ and $y_2(t) \to 0$. as $t \to \infty$. This will be the case if both roots of the characteristic equation $r^2 + ar + b = 0$ have negative real parts.

[44] For all values of A and B

Now these two roots satisfy $r_1 r_2 = b$ and $r_1 + r_2 = -a$. A necessary and sufficient condition for asymptotic stability is therefore that $a, b > 0$.[45]

[45] The roots r_1 and r_2 are strictly below zero if and only if $b > 0$ and $a > 0$. To see this, observe that $b > 0$ and $a > 0$ are necessary if $r_1, r_2 < 0$. Now assume that $a, b > 0$. Then either $r_1, r_2 > 0$ or $r_1, r_2 < 0$ as their product is strictly larger than zero. Also, one of the two must be strictly negative as their sum is strictly negative. Conclude that $r_1, r_2 < 0$.

Systems of linear differential equations

So far we have considered finding one unknown function $y(t)$ to satisfy a single differential equation. Many dynamic economic models involve several unknown functions that satisfy a number of simultaneous differential equations. We will mainly focus on the case where there are two unknowns and two equations

$$\begin{aligned} x' &= f(t, x, y), \\ y' &= g(t, x, y). \end{aligned}$$

In economic models that lead to systems of this type, the solutions $x(t)$ and $y(t)$ are called **state variables**. These variables characterize some economic system at a certain point in time. Usually, the start of the system $(x(t_0), y(t_0))$ is known and the future development is then uniquely determined by the system of differential equations. Systems of this type may exhibit very complicated behaviour.

In special cases the system consists of linear equations,

$$\begin{aligned} x' &= a_{11}x + a_{12}y + b_1(t), \\ y' &= a_{21}x + a_{22}y + b_2(t). \end{aligned}$$

were all $a_{i,j}$ are real valued numbers. Assume $a_{12} \neq 0$.[46] Differentiating the first equation gives,

[46] Otherwise we could first solve $x(t)$ and then substitute in the second to solve $y(t)$.

$$\begin{aligned} x'' &= a_{11}x' + a_{12}y' + b_1'(t), \\ &= a_{11}x' + a_{12}a_{21}x + a_{12}a_{22}y + b_1'(t) + a_{12}b_2(t), \\ &= a_{11}x' + a_{12}a_{21}x + a_{22}(x' - a_{11}x - b_1(t)) + b_1'(t) + a_{12}b_2(t). \end{aligned}$$

So,

$$x'' - (a_{11} + a_{22})x + (a_{11}a_{22} - a_{12}a_{21})x = a_{12}b_2(t) - a_{22}b_1(t) + b_1'(t),$$

which is a linear second order differential equation.[47] We already

[47] Observe that the first coefficient is the trace of the matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

The second coefficient is the determinant of the matrix A.

know how to solve this for $x(t)$. The solution for $y(t)$ is then obtained by substituting the solution of $x(t)$ into the second equation.

However, there is another (sometimes easier) way to solve the system. For simplicity, consider the homogeneous system,

$$x' = a_{11}x + a_{12}y,$$
$$y' = a_{12}x + a_{22}y.$$

Let us see if a choice $x = ce^{\lambda t}, y = de^{\lambda t}$ can give a solution. We have,

$$c\lambda e^{\lambda t} = a_{11}ce^{\lambda t} + a_{12}de^{\lambda t},$$
$$d\lambda e^{\lambda t} = a_{12}ce^{\lambda t} + a_{22}de^{\lambda t}.$$

Deleting common coefficients $e^{\lambda t}$ gives the conditions,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \lambda \begin{bmatrix} c \\ d \end{bmatrix}.$$

This shows that λ is an eigenvector of the matrix[48]

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

and $\begin{bmatrix} c & d \end{bmatrix}$ is an eigenvector of this eigenvalue. The eigenvalues are a solution to the characteristic equation,[49]

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0,$$
$$\leftrightarrow \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0.$$

which is the same as the characteristic equation for the second order differential equation derived above. If the two eigenvalues λ_1, λ_2 are distinct and real with eigenvectors $\begin{bmatrix} \alpha_1 & \beta_1 \end{bmatrix}$ and $\begin{bmatrix} \alpha_2 & \beta_2 \end{bmatrix}$, then the general solution is,

$$x(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t},$$
$$y(t) = \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t}.$$

Here, restrictions on the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ are obtained by substituting the solutions into the differential equation.

If the two eigenvalues are identical with eigenvectors $\begin{bmatrix} \alpha_1 & \beta_1 \end{bmatrix}$ and $\begin{bmatrix} \alpha_2 & \beta_2 \end{bmatrix}$, we obtain the solutions,

$$x(t) = (\alpha_1 + \alpha_2 t)e^{rt},$$
$$y(t) = (\beta_1 + \beta_2 t)e^{rt}.$$

Again, restrictions on the values of the coefficients are obtained by substituting into the differential equations.

[48] For a square matrix A, λ is an eigenvalue if there exists a non-zero vector x such that $Ax = \lambda x$. In this case x is called an eigenvector.

[49] Indeed, $Ax = \lambda x$ implies $[A - \lambda I]x = 0$. So the determinant of $[A - \lambda I]$ should be zero. This gives the condition,

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{1,2}a_{2,1} = 0.$$

As an example, consider the system,

$$\begin{cases} y_1' = 2y_1 + y_2, \\ y_2' = y_1 + 2y_2 \end{cases}$$

The eigenvalues of the matrix A are $\lambda = 1$ and $\lambda = 3$. This gives as solutions is $y_1(t) = \alpha_1 e^t + \alpha_2 e^{3t}$ and $y_2(t) = \beta_1 e^t + \beta_2 e^{3t}$. Substituting in the system gives,

$$\begin{cases} \alpha_1 e^t + 3\alpha_2 e^{3t} = 2\alpha_1 e^t + 2\alpha_2 e^{3t} + \beta_1 e^t + \beta_2 e^{3t}, \\ \beta_1 e^t + 3\beta_2 e^{3t} = \alpha_1 e^t + \alpha_2 e^{3t} + 2\beta_1 e^t + 2\beta_2 e^{3t} \end{cases}$$

Equating exponents gives,

$$\alpha_1 = 2\alpha_1 + \beta_1,$$
$$3\alpha_2 = 2\alpha_2 + \beta_2,$$
$$\beta_1 = \alpha_1 + 2\beta_1,$$
$$3\beta_2 = \alpha_2 + 2\beta_2.$$

This gives $\alpha_1 = -\beta_1$ and $\alpha_2 = \beta_2$, so

$$y_1(t) = \alpha_1 e^t + \alpha_2 e^{3t},$$
$$y_2(t) = -\alpha_1 e^t + \alpha_2 e^{3t}.$$

Finally, if the eigenvalues $\lambda_1 = r + i\theta$ and $\lambda_2 = r - i\theta$ are imaginary with, the resulting solutions are of the form,

$$x(t) = e^{rt}(\alpha_1 \sin\theta t + \alpha_2 \cos\theta t),$$
$$y(t) = e^{rt}(\beta_1 \sin\theta t + \beta_2 \cos\theta t).$$

Consider the linear system with constant coefficients,

$$x' = a_{11}x + a_{12}y + b_1,$$
$$y' = a_{12}x + a_{22}y + b_2.$$

The **stationary points** or **steady state** of this system is determined by the equations,

$$0 = a_{11}x + a_{12}y + b_1,$$
$$0 = a_{12}x + a_{22}y + b_2.$$

If the matrix of a coefficients has non-zero determinant, then this system has a unique solution (x^*, y^*) which is a equilibrium point. In general, the equilibrium point will be globally asymptotically stable if every solution tends to the equilibrium point as $t \to \infty$.

This will be the case if,

$$a_{11} + a_{22} < 0,$$
$$a_{11}a_{22} - a_{12}a_{21} > 0.$$

In other words, the trace of the matrix is negative and the determinant is positive[50] We have the following qualitative features,

1. If both eigenvalues have negative real parts then the equilibrium is globally stable.

2. If both eigenvalues have positive real parts, then the equilibrium is a source and all paths will move away from the equilibrium point.

3. If eigenvalues are real with opposite signs (say $\lambda_1 < 0 < \lambda_2$), then the equilibrium is called a saddle point. In principle there will only be one type of paths that will lead to the equilibrium (the one where the coefficient with $e^{\lambda_2 t}$ is zero). All other starting points will move away from the equilibrium. This kind of equilibrium is encountered frequently in economics[51]

4. If the eigenvalues are purely imaginary, then the equilibrium is called a centre. All paths 'circle' around the equilibrium.

Let us go back to the nonlinear system, and assume that it is autonomous[52]

[50] This is the same as requiring that both eigenvalues have negative real parts.

[51] There is only one path towards the steady state, the optimal one.

[52] The variable t does not occur independently in the dynamic system.

$$x' = f(x, y),$$
$$y' = g(x, y).$$

Assume that there is a unique point (x_s, y_s) where $f(x_s, y_s) = 0 = g(x_s, y_s)$. Then the system can be linearized around this stationary point by taking Taylor expansions.

$$x' \approx f(x_s, y_s) + f_x(x_s, y_s)(x - x_s) + f_y(x_s, y_s)(y - y_s),$$
$$y' \approx g(x_s, y_s) + g_x(x_s, y_s)(x - x_s) + g_y(x_s, y_s)(y - y_s).$$

However, this reduces to,

$$x' = f_x(x_s, y_s)(x - x_s) + f_y(x_s, y_s)(y - y_s),$$
$$y' = g_x(x_s, y_s)(x - x_s) + g_y(x_s, y_s)(y - y_s).$$

Often such linearised system is analysed in order to say something qualitatively of the equilibrium in a neighbourhood of the stationary point. The equilibrium will be (locally) stable if $f_x + g_y < 0$ and $f_x g_y - f_y g_x > 0$.

Phase Plane analysis

EVEN WHEN EXPLICIT solutions are unavailable, geometric arguments can still shed some light on the structure of the solutions of autonomous systems of differential equations. Consider the system,

$$x' = f(x, y),$$
$$y' = g(x, y).$$

A stable state (a, b) where $f(a, b) = g(a, b) = 0$ is usually determined by the intersection of two curves $f(x, y) = 0$ and $g(x, y) = 0$ in the $x - y$ plane. These two paths are called the nullclines of the system.

A phase plane starts by drawing these nullclines. The two nullclines divide the plane in several regions. For each region, depending on where you are above or below or to the left or right of the nullclines, you can determine the sign of x' and y'.[53] These directions can be indicated by arrows in the plane.

[53] The direction of x and y over time.

Let us illustrate this by an example. Consider the following nonlinear system:

$$k' = \sqrt{k} - c,$$
$$c' = c[16 - 4k]$$

We can construct a diagram with k on the horizontal axis and c on the vertical axis. Each point in the space represents the position of the system (k, c) at a given moment in time. If we want to see what the position of the economy will be at the "next instant" we can represent the dynamics with arrows that point in the direction of motion. To construct the phase diagram, follow these steps:

1. Plot the locus of points for which k' equals 0. This is given by the equation $c = \sqrt{k}$.

2. Analyse the dynamics of k in each of the two regions generated by the $k' = 0$ locus
   - for $c < \sqrt{k}$, $k' > 0$, hence arrows point east
   - for $c > \sqrt{k}$, $k' < 0$, so arrow point west.

3. repeat the procedure for c'.
   - the $c' = 0$ locus is given by the equation $k = 10$, a vertical line
   - for $k > 4$, we have that $c' < 0$ so arrows point south
   - for $k < 4$, we have that $c' > 0$ so arrows point up.

4. combine the dynamics for k and c
   - the steady state is the point at which the $k' = 0$ and $c' = 0$ loci cross, a condition that corresponds to $k = 4$ and $c = \sqrt{4} = 2$.
   - the dynamics of the system (the arrows) determine the stability of the system. In this case, we have a saddle point equilibrium.

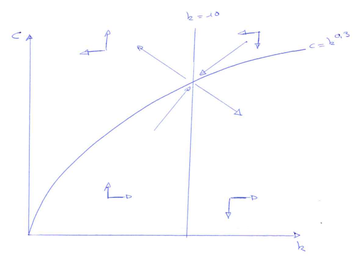

Figure 9: Phase plane diagram

Exercises

Separable differential equations

1. $3y^2y' = t$ with $y(0) = 2$. (Sol: $y(t) = \left[t^2/2 + 8\right]^{1/3}$)

2. $y' = y^3/t^3$ with $y(1) = 2$ (Sol: $y(t)^2 = \frac{4t^2}{4-3t^2}$).

3. $y' = t^3/y^3$ with $y(1) = 2$ (Sol: $y(t)^4 = t^4 + 15$).

4. $(1 + t^2)y' + ty = 0$ with $y(1) = 2$ (Sol: $y = 2\sqrt{\frac{2}{(1+t^2)}}$.).

Homogeneous differential equations

1. $y' = -\frac{t^2+y^2}{2ty}$ (Sol: $t_0^3 + 3y_0^2 t_0 = t^3 + 3ty^2$).

2. $y' = \frac{y}{t} + e^{y/t}$ (Sol: $y = -t \ln(\ln(v_0/t))$).

Linear differential equations

1. $\mathbf{y} + 5y = 15$ (Sol: $y(t) = 3 + Ce^{-5t}$).

2. $y' = y + e^t$ (Sol: $y(t) = te^t + Ce^t$).

3. $y' + t^2y = 3t^2$ (Sol: $y = 3 - (3 - y_0)e^{t_0^3/3 - t^3/3}$.).

4. $y' + 6y = e^{-t}$ with $y(0) = 6/7$ (Sol: $y(t) = \frac{23}{35}e^{-6t} + \frac{e^{-t}}{5}$).

Bernoulli differential equation

1. $y' + y/t = y^3$ (Sol: $y(t)^2 = \frac{1}{Ct^2+2t}$).

Exact differential equations

1. $y^2 + 2yty' = 0$ (Ans: $y^2t = C$).

2. $(y + 3t^2) + (t + 2y)y' = 0$ (Ans: $yt + t^3 + y^2 = C$).

3. $3yt^2 + 2(t^3 + 1)y' = 0$ (this is not an exact equation, try t, y and y^2 as integrating factors) (Ans: $y^2t^3 + y^2 = C$)

Linear differential equations of order > 1

1. $y'' - 2y' + y = 6te^t$ (Ans: $y(t) = Ae^t + Bte^t + t^3e^t$.

2. $y'' - 4y' + 13y = 10\cos 2t + 25\sin 2t$ (Sol: $y(t) = e^{2t}(A\sin 3t + B\cos 3t) + \sin 2t + 2\cos 2t$.).

3. $y'' - y = e^{3t}$ (Sol: $y(t) = Ae^t + Be^{-t} + e^{3t}/8$).

4. $y^{(3)} - 3y'' - 4y' + 12y = e^{3t}$ (Sol: $y(t) = Ae^{3t} + Be^{-2t} + Ce^{2t} + (1/5)te^{3t}$.).

5. $y^{(3)} - y'' + y' - y = 2e^t$ (Sol: $y(t) = Ae^t + B\sin t + C\cos t + te^t$).

Systems of differential equations

1. $\begin{cases} x' = x + 12y - 60, \\ y' = -x - 6y + 36 \end{cases}$ (Sol: $x(t) = -3Ae^{-3t} + 4Be^{2t} + 12; y(t) = Ae^{-3t} - Be^{2t} + 4$).

2. $\begin{cases} x' = 3x + y, \\ y' = 3y, \\ z' = 2z \end{cases}$ (Sol: $x(t) = Ae^{3t} + Bte^{3t}, y(t) = Be^{3t}, z(t) = Ce^{2t}$.)

Draw a phase diagram, find the equilibrium points and draw the direction of motion of the following system:

- $\begin{cases} x' = y, \\ y' = -2x - y \end{cases}$
- $\begin{cases} K' = aK - bK^2 - C, \\ C' = w(a - 2bK)C \end{cases}$
- $\begin{cases} x' = y, \\ y' = x \end{cases}$
- $\begin{cases} x' = x + y, \\ y' = x - y \end{cases}$
- $\begin{cases} x' = x(y - x/2 - 2), \\ y' = y(1 - y/2x) \end{cases}$

Intro to dynamic optimization

Static optimization problems require you to find the value of one or several variables possibly subject to one or several equality or inequality constraints that maximize a certain function.

Many economic processes, however, are by nature dynamic: "decisions" made at some time t depend on those made before and will influence those made after. Dynamic optimization requires to find the path of one or several variables, eventually subject to constraints in such a way to maximize or minimize a given functional. For example, we could be interested in the optimal path between two points in 2 dimensional space if each admissible path is associated with some cost which is a function not only of the distance, but also of the topography (the objective being to minimize total cost).

Basically, a dynamic optimization problem consists of

1. An initial and terminal point.
2. A set of admissible paths (from the initial to the terminal point).
3. A set of values (costs, profits, ...) associated to each admissible path.
4. An objective (to be maximized or minimized).

However, there are several variations on this.

In the standard problem, both terminal time and terminal value are given. In some cases, the terminal time is given but not the value of the path at the terminal time. In some cases, the terminal time is free but the value at the terminal time is fixed. In some cases, both terminal time and terminal values are free but they have to satisfy some restrictions.

In all these situations, the planner has one more degree of freedom than in the fixed terminal time-value case. We will see that an extra condition will be required to be able to distinguish the optimal path from the other admissible paths. Such conditions are known as **transversality** conditions: how the optimal path crosses (transverses) the terminal line or curve.

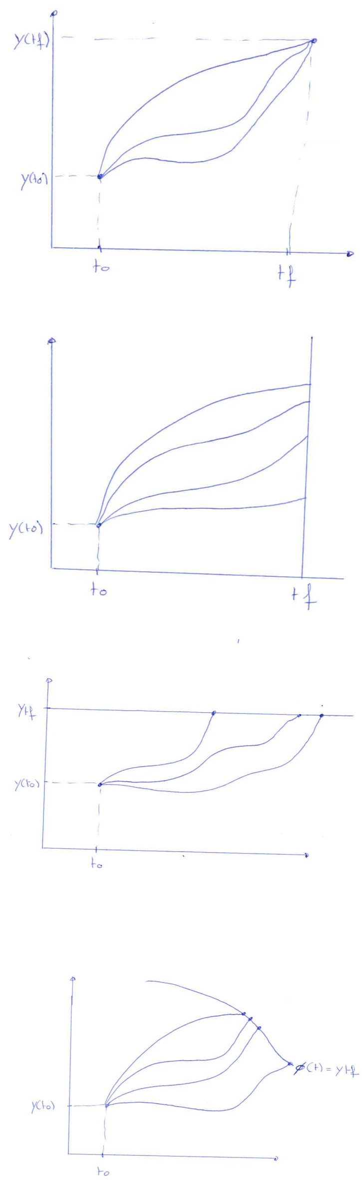

Figure 10: Top: both initial and terminal values are given. Second: terminal time fixed but value is variable. Third: terminal value fixed but time is variable. Bottom both terminal time and value are variable but restricted to lie on some curve.

A path with initial time t_0 and final time t_f can be described by a function $y : [t_0, t_f] \to \mathbb{R}$ let $\mathcal{T}$ be the set of all admissible paths $\mathcal{T}$. A value function is then a mapping from $\mathcal{T}$ onto $\mathbb{R}$.

$$V : \mathcal{T} \to \mathbb{R} : y(t) \mapsto V(y(t)).$$

An optimal path $y^*(t)$ defined on the interval $[t_0, t_f]$ is a path which maximizes (or minimizes) $V[y(t)]$.[54]

In a discrete framework, the latter total value is quite naturally the sum of the values associated to each period. In the continuous time framework, we assume that $V[y(t)]$ takes the form of an integral from t_0 to t_f,[55]

$$V[y(t)] = \int_{t_0}^{t_f} (\text{value of an infinitsemal arc})dt = \int_{t_0}^{t_f} F(t, y(t), y'(t))dt.$$

Here, as usual, t is time, $y(t)$ is the state at time t and $y'(t)$ is the direction in which the path proceeds.[56]

Also, we can consider the problem $V[y(t)] = \int_0^T F(t, y(t), \dot{y}(t))dt + G(t_f, y(t_f))$ which can be brought back to the standard problem.

[54] $V[y(t)]$ is the total value of $y(t)$.

[55] An infinite sum of infinitesimal values.

[56] Other possible forms are $V[y(t)] = \psi(t_f, y(t_f))$. Here the criterion does not depend on the intermediary positions. Writing $z(t) = \psi(t, y(t))$ with $z(t_0) = 0$, we obtain $\psi(t_f, y(t_f)) = \int_{t_0}^{t_f} z'(t)dt$.

THERE ARE VARIOUS approaches to solve dynamic optimization problems. Which solution method is most appropriate often depends on the problem under consideration.

1. **Calculus of Variation**[57] The fundamental problem of the calculus of variations is the following

$$\max V[y(t)] = \int_{t_0}^{t_f} F[t, y(t), y'(t)]dt \text{ with } y(t_0) = y_0, y(t_f) = y_f.$$

   The implicit assumption here is that all functions are at least C^1.

2. **Optimal control** Optimal control problems are a generalization of calculus of variation problems. Optimal control problems have three types of variables: t, which denotes time, $y(t)$, which is the state variable and $u(t)$, which is a control variable. In order to unambiguously determine the state variable path $y(t)$, an equation linking $y(t)$ and $u(t)$ is necessary. This is the so-called equation of motion or state equation,

$$y'(t) = f(t, y(t), u(t)).$$

   This equation describes how, at each point in time, the control variable $u(t)$ drives the state variable $y(t)$. As such, the optimal path for $u(t)$ determines the optimal path for $y(t)$. The classical problem is

$$\max V[u(t)] = \int_{t_0}^{t_f} F[t, y(t), u(t)]dt$$
$$\text{subject to } y'(t) = f(t, y(t), u(t)); y(t_0) = y_0, y(t_f) = y_f.$$

[57] < 17th century, Newton (1687), Bernoulli (Jean and Jacques) (1696, 1697), Lagrange (1760), Legendre (1786), Jacobi (1837), Weierstrass (1870). The original problem that started the development of this method is the following: which type of surface of revolution would encounter the least resistance when moving through some resisting medium (= surface of revolution with the minimum area).

Optimal control problems are more general than calculus of variations problems.[58] Optimal control problems are solved by the Maximum principle.[59] It allows further restrictions on $u(t)$ such as $u(t) \in \mathcal{U}$ with $\mathcal{U}$ convex bounded and closed.

3. **Dynamic programming**[60] This is another approach of optimal control embedding the problem into a larger class of problems. It focuses on the optimal value function $V[y^*(t)]$, rather than on $u^*(t)$ and $y^*(t)$. In the continuous time framework, solving this type of problems requires solving partial differential equations.[61]

[58] Indeed, setting $y'(t) = u(t)$ in the turns the optimal control problem into a calculus of variations problem.

[59] Potryagin, 1962.

[60] Bellman, (1957).

[61] The dynamic programming approach is very popular for discrete time dynamic optimization problems.

The calculus of variations

In this chapter, we deal with the calculus of variations. In particular, we look at the following problem

$$\max V[y(t)] = \int_{t_0}^{t_f} F[t, y(t), \dot{y}(t)]dt,$$
$$\text{s.t. } y(t_0) = y_0; y(t_f) = y_f,$$

where $y(t)$ is restricted to be C^1 and F is assumed to be C^2.

An **absolute maximum** of $V[y(t)]$ occurs at a path $y^*(t)$ if for all admissible paths $y(t)$, $V[y(t)] \leq V[y^*(t)]$. A **relative or local maximum** of $V[y(t)]$ occurs at $y^*(t)$ if there is a $\rho > 0$ such that for all admissible paths $y(t)$ with $|\bar{y}(t) - y^*(t)| < \rho$ for all $t \in [0, T]$: $V[y(t)] \leq V[y^*(t)]$. From now on, we will focus on local optima.

The main idea behind the calculus of variations[62] is to look at small deviations from the optimal path. The underlying idea is that if the path is optimal, then small deviations should not increase the value function.

[62] And many other solution methods for optimization problems that rely on first order conditions.

Let us assume that $y^*(t)$ is a solution of to the dynamic optimization problem[63] Consider the path,

[63] Here, a local optimum.

$$y(t) = y^*(t) + \varepsilon\eta(t),$$

where we assume,

$$\eta(t) \in C^1[t_0, t_f],$$
$$\eta(t_0) = \eta(t_f) = 0.$$

We also require that $\varepsilon \in \mathbb{R}$ and $|\varepsilon|$ is sufficiently small such that $V[y^*(t)] \geq V[y(t)]$ for all considered ε.[64] The conditions on $\eta(t)$ guarantee that $y(t)$ is C^1 and satisfies the terminal conditions $y(t_0) = y_0$ and $y(t_f) = y_f$.[65]

[64] So we require $y(t)$ to be in a local neighbourhood of $y^*(t)$.

[65] Indeed, $y(t_0) = y^*(t_0) + \varepsilon\eta(t_0) = y^*(t_0) = y_0$ and $y(t_1) = y^*(t_1) + \varepsilon\eta(t_1) = y^*(t_1) = y_1$.

Obviously $\lim_{\varepsilon\to 0} y(t) = y^*(t)$. We can write[66]

$$V[y(t)] = V[y^*(t) + \varepsilon\eta(t)] \equiv \tilde{V}[\varepsilon].$$

[66] In other words, we fix $y^*(t)$ and $\eta(t)$ and look at $V[y(t)]$ as a variable of ε alone.

Now, we have that for $|\varepsilon|$ sufficiently small $V[\varepsilon] \leq V[0]$. In other words $V[\varepsilon]$ obtains a local interior maximum at $\varepsilon = 0$. Then, the first order condition requires that

$$\frac{\partial V[0]}{\partial \varepsilon} = 0.$$

By eliminating ε and $\eta(t)$ from the first order condition we obtain the so-called **Euler-Lagrange equation**. Towards this end, let us start by writing $V[\varepsilon]$ out in full:

$$V[\varepsilon] = \int_{t_0}^{t_f} F[t, y^*(t) + \varepsilon\eta(t), y'^*(t) + \varepsilon\eta'(t)]dt.$$

The first order condition gives,[67]

$$\frac{\partial V[0]}{\partial \varepsilon} = \int_{t_0}^{t_f} \left[F_y[t, y^*(t), y'^*(t)]\eta(t) + F_{y'}[t, y^*(t), y'^*(t)]\eta'(t)\right] dt = 0.$$

Let us abuse notation and write $F(t) = F(t, y^*(t), y'^*(t))$. First, we integrate the second term by parts,

$$\int_{t_0}^{t_f} F_{y'}(t)\eta'(t)dt = \left[F_{y'}(t)\eta(t)\right]_{t_0}^{t_f} - \int_{t_0}^{t_f} \eta(t)\frac{d}{dt}\left[F_{y'}(t)\right] dt = 0.$$

The first order condition becomes,

$$\int_0^{t_f} \left[F_y(t) - \frac{d}{dt}F_{y'}(t)\right] \eta(t)dt + F_{y'}(t_f)\eta(t_f) - F_{y'}(t_0)\eta(t_0) = 0.$$

The last two terms are zero.[68] This gives the condition,

$$\int_{t_0}^{t_f} \left[F_y(t) - \frac{d}{dt}F_{y'}(t)\right] \eta(t)dt = 0.$$

This has to hold for all $\eta(t) \in C^1[t_0, t_f]$ with $\eta(t_0) = \eta(t_f) = 0$. The following lemma will be useful.

Lemma 3 (Fundamental lemma of the calculus of variations)**.** *Let $M(t) \in C^1[t_0, t_f]$ such that $\int_{t_0}^{t_f} M(t)\eta(t)dt = 0$ for all $\eta(t) \in C^1[t_0, t_f]$ with $\eta(t_0) = \eta(t_f) = 0$. Then $M(t) = 0$ for all $t \in [t_0, t_f]$.*

Proof. Assume, towards a contradiction, that there is some $\hat{t} \in [t_0, t_f]$ such that $M(\hat{t}) \neq 0$. Without loss of generality $M(\hat{t}) > 0$.[69] As $M(t)$ is C^1 on $[t_0, t_f]$, there exists an interval $]\alpha, \beta[\subseteq [t_0, t_f]$ such that for all $t \in]\alpha, \beta[$, $M(t) > 0$. Define,

$$\eta(t) = \begin{cases} 0 \text{ if } t \in [0, T] \setminus]\alpha, \beta[, \\ (t-\alpha)^2(t-\beta)^2 \text{ for } t \in]\alpha, \beta[\end{cases}$$

The function $\eta(t) \in C^1[t_0, t_f]$,[70] $\eta(t_0) = \eta(t_f) = 0$ and $\eta(t) > 0$ for all $t \in]\alpha, \beta[$. As such,

$$\int_{t_0}^{t_f} \eta(t)M(t)dt > 0,$$

a contradiction. Conclude that $M(t) = 0$ for all $t \in [t_0, t_f]$. □

[67] Remember Leibniz' rules.

[68] Because $\eta(t_f) = \eta(t_0) = 0$.

[69] Try also the proof when $M(\hat{t}) < 0$.

[70] Show this!

Using above lemma, the see that,

$$F_y(t, y^*, y'^*) - \frac{d}{dt} F_{y'}(t, y^*, y'^*) = 0.$$

This important equation is called the **Euler-Lagrange equation**.

The Euler-Lagrange equation $F_y - \frac{d}{dt} F_{y'} = 0$ can be rewritten as,

$$F_y - F_{y't} - F_{y'y} y' - F_{y'y'} y'' = 0.$$

Unless $F_{y'y'} = 0$, this is a second-order differential equation. Its general solution, contains two arbitrary constants. The boundary values $y(t_0) = y_0$ and $y(t_f) = y_{t_f}$ should allow you determine the exact solution, but this is not always the case. The solutions of the Euler-Lagrange equation, which are admissible are candidates for maxima and minima of the problem.

LET US HAVE a look at several examples.

1. Consider the problem,

$$V[y(t)] = \int_0^2 (12ty + y'^2) dt \text{ with } y(0) = 0, y(2) = 8.$$

The Euler-Lagrange condition gives,

$$\begin{aligned} F_y - \frac{d}{dt} F_{y'} &= 0, \\ \leftrightarrow 12t - \frac{d}{dt}\left[2y'\right] &= 0, \\ \leftrightarrow 12t - 2y'' &= 0, \\ \leftrightarrow y'' &= 6t, \\ \rightarrow y' &= 3t^2 + A, \\ \rightarrow y &= t^3 + At + B. \end{aligned}$$

The initial and terminal conditions give $y(0) = B = 0$ and $y(2) = 2^3 + 2A = 2^3 + 2A = 8$ or $A = 0$. As such, the solution is $y(t) = t^3$.

2. As a second example, let us try to find the extremals of

$$V[y(t)] = \int_1^5 [3t + (y')^{1/2}] dt \text{ with } y(1) = 3; y(5) = 7.$$

Euler-Lagrange gives,

$$\begin{aligned} F_y - \frac{d}{dt} F_{y'} &= 0, \\ \leftrightarrow -\frac{d}{dt}\left[0.5(y')^{-1/2}\right] &= 0, \\ \leftrightarrow -0.25(y'')^{-3/2} &= 0, \\ \leftrightarrow y'' &= 0, \\ \rightarrow y' &= A, \\ \rightarrow y &= At + B. \end{aligned}$$

The initial conditions give $y(1) = A + B = 3$ and $y(5) = 5A + B = 7$. So $A = 1$ and $B = 2$. The solution is $y = t + 2$.

3. As a third example consider the value function,

$$V[y(t)] = \int_0^5 [t + y^2 + 3y'] dt \text{ with } y(0) = 0; y(5) = 3.$$

The Euler-Lagrange gives,

$$\begin{aligned} &F_y - \frac{d}{dt}F_{y'} = 0, \\ \leftrightarrow &2y - \frac{d}{dt}[3] = 0, \\ \leftrightarrow &y = 0. \end{aligned}$$

There is no extremal solution here as $y(5) = 3$ is not satisfied.

4. As a final example, consider,

$$V[y(t)] = \int_0^T y' dt \text{ with } y(0) = \alpha; y(t_f) = \beta.$$

The Euler Lagrange gives,

$$\begin{aligned} &F_y - \frac{d}{dt}F'_y = 0, \\ \leftrightarrow &0 - \frac{d}{dt}[1] = 0, \\ \leftrightarrow &0 = 0. \end{aligned}$$

The latter equation is satisfied by any admissible path. There are an infinite number of extremals here.

THERE ARE SEVERAL special cases worth mentioning. If $F(t, y, y')$ is linear in y', the Euler-Lagrange can be simplified,

$$\begin{aligned} &F_y - \frac{d}{dt}\left[F_{y'}\right] = 0, \\ \leftrightarrow &F_{y'y'}y'' + F_{y'y}y' + F_{y't} - F_y = 0, \\ \leftrightarrow &F_{y'y}y' + F_{y't} - F_y = 0 \end{aligned}$$

This is no longer a second order equation.

Next, if $F = F(t, y')$ then F does not explicitly depend on y, so $F_y = 0$.The Euler Lagrange becomes $\frac{d}{dt}F_{y'} = 0$ or $F_{y'} = C$, a constant, this is also a first order differential equation.

Third, If $F = F(y, y')$ then F does not explicitly depend on t. Then we have,

$$\begin{aligned} \frac{d}{dt}\left[F - y'F_{y'}\right] &= F_y y' + F_{y'}y'' - y''F_{y'} - y'F_{yy'}y' - y'F_{y'y'}y'', \\ &= y'\left[F_y - \frac{d}{dt}\left[F_{y'}\right]\right] = 0. \end{aligned}$$

From this, we see that $F - y'F_{y'} = C$ gives the solution to the Euler equation.

Finally, if $F = F(t, y)$ then F does not explicitly depend on y'. In this case, the condition becomes $F_y = 0$ which is generally not a differential equation. In general, this condition will not satisfy the initial and terminal conditions.

Find $y(t)$ that minimizes the surface of revolution around axis t with $y(\alpha) = A$ and $y(\beta) = Z$.

$$V[y(t)] = 2\pi \int_\alpha^\beta y\sqrt{1 + (y')^2}dt.$$

The Euler-Lagrange condition gives

$$\begin{aligned} &F - y'F_{y'} = C, \\ \leftrightarrow &y\sqrt{1 + (y')^2} - y'y\frac{y'}{\sqrt{1 + (y')^2}} = C, \\ \leftrightarrow &y(1 + (y')^2 - (y')^2) = C\sqrt{1 + (y')^2}, \\ \leftrightarrow &y^2 = c^2(1 + (y')^2), \\ \leftrightarrow &y^2 - c^2 = c^2(y')^2, \\ \rightarrow &y' = \frac{1}{c}\sqrt{y^2 - c^2}, \\ \rightarrow &\int \frac{cdy}{\sqrt{y^2 - c^2}} = t + K, \\ \rightarrow &c\ln\left(\frac{y + \sqrt{y^2 - c^2}}{c}\right) = t + K, \\ \rightarrow &\frac{y + \sqrt{y^2 - c^2}}{c} = e^{\frac{t+K}{c}}, \\ \rightarrow &\sqrt{y^2 - c^2} = ce^{\frac{t+K}{c}} - y, \\ \rightarrow &y^2 - c^2 = c^2e^{\frac{2(t+K)}{c}} - 2ce^{\frac{t+K}{2}}y + y^2, \\ \rightarrow &y = \frac{c}{2}\left(e^{\frac{t+K}{2}} + e^{-\frac{t+K}{c}}\right) \end{aligned}$$

THE PROBLEM CAN be generalized in several direction.

- Several state variables:

$$V[y_1(t), \dots, y_n(t)] = \int_0^T F[t, y_1(t), \dots, y_n(t), y_1'(t), \dots, y_n'(t)]dt.$$

  with initial and terminal conditions for each $y_i(t)$.

  It can be easily shown that the Euler-Lagrange equation, the first order necessary optimization condition, becomes for this situation, a system of n Euler-Lagrange equations,

$$F_{y_i} - \frac{d}{dt}\left[F_{y_i'}\right] = 0.$$

Consider,

$$V[y(t), z(t)] = \int_0^T (y + z + (y')^2 + (z')^2)dt.$$

We have $F(t, y(t), z(t), y'(t), z'(t)) = y + z + (y')^2 + (z')^2$ so

$$1 - 2y'' = 0,$$
$$1 - 2z'' = 0$$

This gives $y = (1/4)t^2 + At + B$ and $z = (1/4)t^2 + Ct + D$, where the coefficients are determined by the initial and terminal conditions.

- Presence of higher order derivatives

$$V[y(t)] = \int_0^T F[t, y, y', y'', \dots, y^{(n)}]dt.$$

  Plus initial and terminal conditions for $y, y', y'', \dots, y^{(n-1)}$. This case can be brought back to the previous one by letting $z = y'$, $u = y''$, ...

  Another necessary condition for an extremal is the so-called Euler Poisson equation,

$$F_y - \frac{d}{dt}\left[F_{y'}\right] + \frac{d^2}{dt^2}\left[F_{y''}\right] + \dots + (-1)^n \frac{d^n}{dt^n}\left[F_{y^{(n)}}\right] = 0.$$

  This equation is in general a differential equation of order $2n$ in $y(t)$. There are $2n$ arbitrary constants in the general solution which are to be determined via the initial and terminal conditions.

Transversality conditions

THE EULER-LAGRANGE EQUATION is, in general, a second order differential equation. Solving it leads to a solution with two degrees of freedom. The initial conditions $y(t_0) = y_0$ and $y(t_f) = y_{t_f}$ then allows to find the exact solution.

If parts of the initial conditions are missing,[71] one can use so-called **transversality conditions** to replace the missing initial and terminal conditions.

[71] For example, if the initial or terminal point is not fixed.

- Fixed terminal time t_f but free terminal values y_{t_f}
  Assume that initial and terminal periods t_0 and t_f are given but the values $y(t_0) = y_0$ and $y(t_f) = y_f$ are free.

Let $V[y(t)] = \int_{t_0}^{t_f} F[t, y, y']$. By letting

$$y(t) = y^*(t) + \varepsilon\eta(t)$$

as before where $y^*(t)$ is an extremal of the problem and $\eta(t) \in C^1[t_0, t_f]$ with no conditions on $\eta(0)$ or $\eta(t_f)$. We have

$$V[\varepsilon] = \int_{t_0}^{t_f} F[t, y(t), \dot{y}(t)]dt.$$

The first order necessary conditions are

$$\int_{t_0}^{t_f} \left[F_y(t)\eta(t) + F_{y'}(t)\eta'(t))\right] dt = 0.$$

As, before, integrating by parts the second term gives,

$$\int_{t_0}^{t_f} \left[F_y(t) - F_{y'}(t)\right] \eta(t)dt + F_{y'}(t_f)\eta(t_f) - F_{y'}(t_0)\eta(t_0) = 0.$$

Now, this condition has to hold for all $\eta(t) \in C^1[t_0, t_f]$.[72]

[72] Notice that the final two terms are not necessarily equal to zero.

– Choosing $\eta(t_0) = \eta(t_f) = 0$ provides the standard Euler-Lagrange condition

$$F_y - \frac{d}{dt}\left[F_{y'}\right] = 0.$$

This makes sure that the first term in the first order condition is zero.

– Choosing $\eta(t_0) = 0$ and $\eta(t_f) = 1$, we have the additional condition that,

$$[F_{y'}]_{t_f} = 0.$$

– Choosing $\eta(t_0) = 1$ and $\eta(t_f) = 0$, we have the additional condition

$$[F_{y'}]_{t_0} = 0.$$

These two additional conditions are called the **natural limit conditions**.[73] The condition $[F_{y'}]_{t_f} = 0$ roughly means that a slight change of course at the last moment cannot improve the objective function.

[73] If only one of the two boundary values is free, only the corresponding limit condition is necessary.

- Fixed terminal time and bounded terminal value
If the terminal line is vertical (t_f is fixed) but truncated, which means that $y^*(t_f)$ is unknown but $y^*(t_f) \geq y_{\min}$ for some $y_{\min}$. then we have for a maximization,[74]

$$[F_{y'}]_{t_f} \leq 0,$$
$$y^*(t_f) \geq y_{\min},$$
$$(y^*(t_f) - y_{\min})[F_{y'}]_{t_f} = 0.$$

[74] Notice the similarity with the Kuhn-Tucker complementary slackness conditions. If $[F_{y'}]_{t_f} < 0$, you would like to decrease y_{t_f} but as $y_{t_f} = y_{\min}$ this is not allowed.

From a practical point of view, $[F_{y'}]_{t_f} = 0$ is first considered and if the resulting optimal path satisfies $y^*(t_f) \geq y_{min}$ we have found a solution. Otherwise one sets $y^*(t_f) = y_{\min}$ and $y^*(t_f)$ is fixed.

- Restricted terminal value and time
  Assume that the terminal time t_f and terminal value $y(t_f)$ is free but that it is required that $y(t_f) = \phi(t_f)$ for some C^1 function $\phi(t)$. So we have the problem

$$\max_{y(t),t_f} \int_{t_0}^{t_f} F[t, y(t), y'(t)]dt \text{ s.t. } y(t_0) = y_{t_0}, y(t_f) = \phi(t_f).$$

Let $y^*(t)$ be an extremal solution and let $y(t) = y^*(t) + \varepsilon\eta(t)$ then, we can set up the Lagrangian,[75]

$$L(\varepsilon, t_f, \lambda) = \int_{t_0}^{t_f} F[t, y(t), y'(t)]dt - \lambda(\phi(t_f) - y(t_f)).$$

Observe that t_f is a choice variable. The first order conditions give,

$$\begin{aligned}
L_\varepsilon &= \int_{t_0}^{t_f} [F_y(t)\eta(t) + F_{y'}(t)\eta'(t)]dt + \lambda\eta(t_f) = 0, \\
L_{t_f} &= F(t_f) - \lambda(\phi'(t_f) - y'(t_f)) = 0, \\
L_\lambda &= \phi(t_f) - y(t_f) = 0.
\end{aligned}$$

The first condition can be rewritten as,[76]

$$\int_{t_0}^{t_f} [F_y(t) - \frac{d}{dt}F_{y'}(t)]\eta(t)dt + [F_{y'}]_{t_f}\eta(t_f) - [F_{y'}]_{t_0}\eta(t_0) + \lambda\eta(t_f) = 0.$$

From the second first order condition, we get,

$$\lambda = \frac{[F]_{t_f}}{\phi'(t_f) - y'(t_f)}.$$

Substituting into the first condition (and acknowledging that $\eta(t_0) = 0$) gives,

$$\int_{t_0}^{t_f} [F_y(t) - \frac{d}{dt}F_{y'}(t)]\eta(t)dt + \left([F_{y'}]_{t_f} + \frac{[F]_{t_f}}{\phi'(t_f) - y'(t_f)} \right) \eta(t_f) = 0.$$

This has to hold for all admissible paths $\eta(t)$. Choosing $\eta(t_f) = 0$ gives the standard Euler-Lagrange condition.

$$F_y - \frac{d}{dt}F_{y'} = 0.$$

Next choosing $\eta(t_f) \neq 0$ gives the additional two condition,

$$\begin{aligned}
&[F + (\phi' - y')F_{y'}]_{t_f} = 0, \\
&\phi(t_f) - y(t_f) = 0.
\end{aligned}$$

[75] It is also possible to make a detour using the implicit function theorem, but (as we have seen in the constrained optimization part) this leads to the same first order conditions.

[76] Using integration by parts.

- Terminal time free, terminal value fixed
  This would impose the condition $y(t_f^*) = y_f$ so the Lagrangian becomes,

$$L(\varepsilon, t_f, \lambda) = \int_{t_0}^{t_f} F[t, y(t), y'(t)]dt - \lambda(y_f - y(t_f^*)).$$

  Doing the maths, you find, in addition to the Euler-Lagrange conditions, that the following two conditions have to be satisfied.

$$\begin{aligned}
&[F - y'(t_f^*)F_{y'}]_{t_f} = 0,\\
&y(t_f) = y_f.
\end{aligned}$$

- Terminal time bounded, terminal value fixed
  In this case, you have the following two conditions,

$$\begin{aligned}
y(t_f) &= y_{t_f},\\
t_f &\leq t_{\max}.
\end{aligned}$$

  This gives the complementary slackness condition,

$$(t_f^* - t_{\max})[F - y'F_{y'}]_{t_f} = 0.$$

  The way this is solved is to first look at an optimal condition with the additional constraint $[F - y'F_{y'}]_{t_f} = 0$. If this optimal condition gives $t_f \leq t_{\max}$ the optimal solution is found. Else, you solve for the optimal solution with the additional constraint $t_f = t_{\max}$.

In case of several state variables, for example two, the objective function is $F(t, y(t), z(t), \dot{y}(t), \dot{z}(t)]$. Then if t_f is fixed but $y(t_f)$ and $z(t_f)$ are free, we have $[F_{y'}]_{t=t_f} = 0$ and $[F_{z'}]_{t=t_f} = 0$.
If $y(t_f) = \psi(t_f)$ and $z(t_f) = \phi(t_f)$, we have the condition $\Big[F + (\psi' - y')F_{y'} + (\phi' - z')F_{z'}\Big]_{t=t_f} = 0.$
In case derivatives of higher order appear, $F[t, y(t), y'(t), y''(t), \ldots]$ transversality conditions are more complex.

LET US HAVE a look at several examples. For the first, let us find the path $y(t)$ that achieves the shortest distance between $(0, 1)$ and $y = 2 - t$. This is the $y(t)$ that minimizes $\int_0^{t_f} \sqrt{1 + (y')^2}dt$ with $y(0) = 1$ and $y(t_f) = 2 - t_f$.

The Euler-Lagrange condition gives,

$$\begin{aligned}
&F_y - \frac{d}{dt}F_{y'} = 0,\\
\rightarrow &\frac{2y'}{2\sqrt{1 + (y')^2}} = 0,\\
\rightarrow &\sqrt{1 + (y')^2} = C,\\
\rightarrow &(y')^2 = C^2 - 1,\\
\rightarrow &y' = \sqrt{C^2 - 1},\\
\rightarrow &y' = k,
\end{aligned}$$

for some constant k. The general solution is then $y(t) = kt + A$ for some constant of integration A. Also, we have the condition,

$$
\begin{aligned}
&[F + (\phi' - y')F_{y'}]_{t_f} = 0,\\
\rightarrow &\left[\sqrt{1+(y')^2} + (-1-y')\frac{2y'}{2\sqrt{1+(y')^2}}\right]_{t_f} = 0,\\
\rightarrow &\left[\sqrt{1+k^2} + (-1-k)\frac{2k}{2\sqrt{1+k^2}}\right] = 0,\\
\rightarrow &(1+k^2) = (k+1)k,\\
\rightarrow &1 = k.
\end{aligned}
$$

So $y(t) = t + C$ is the general solution. The initial value gives $C = 1$ so we have $y(t) = t + 1$.

As a second example, let us find the extremals of $V[y(t)] = \int_0^{t_f}(ty' + (y')^2)dt$ with $y(0) = 1$ and $y(t_f) = 10$ with t_f free. The Euler-Lagrange condition gives,

$$
\begin{aligned}
&0 - \frac{d}{dt}\left[t + 2y'\right] = 0,\\
\Longleftrightarrow\ &1 + 2y'' = 0,\\
&\rightarrow y'' = -1/2, \rightarrow y' = -t/2 + C, \rightarrow y = -t^2/4 + Ct + D.
\end{aligned}
$$

The transversality condition requires that

$$
\begin{aligned}
&[ty' + (y')^2 - y'(t+2y')]_{t_f} = 0,\\
\Longleftrightarrow\ &[-(y')^2]_{t_f} = 0, \rightarrow y'(t_f) = 0, \rightarrow -t_f/2 + C = 0, \rightarrow C = t_f/2.
\end{aligned}
$$

Also $y(0) = 1$ gives $D = 1$ and since $y(t_f) = 10$ we also have $-t_f^2/4 + t_f^2/2 + 1 = 10$ or $10 = t_f^2/4 + 1$, wich gives $t_f = 6$ and $C = 3$ as solutions.

So far only extremals have been detected. In order to be sure that they are optimal, we will give a sufficient condition for a maximum or a minimum and a necessary second order condition.

As before, let

$$y(t) = y^*(t) + \varepsilon\eta(t),$$

with $y^*(t)$ an extremal of the maximization or minimization problem.

From the first order necessary condition, $\frac{\partial V[0]}{\partial\varepsilon} = 0$, we deduced the Euler-Lagrange equation, a necessary condition[77] By pursuing the reasoning on $V[\varepsilon]$, we obtain second order sufficient conditions for a local maximum or minimum, i.e.,

$$
\begin{aligned}
&\frac{\partial^2 V}{\partial\varepsilon^2} > 0 \text{ (minimum)},\\
&\frac{\partial^2 V}{\partial\varepsilon^2} < 0 \text{ (maximum)}.
\end{aligned}
$$

[77] If the initial conditions are not fixed, then the transversality conditions must be included as an additional necessary condition.

We have

$$\begin{aligned}\frac{\partial^2 V}{\partial \varepsilon^2}[\varepsilon] &= \int_{t_0}^{t_f} \left[\frac{d}{d\varepsilon} F_y(t)\eta(t) + \frac{d}{d\varepsilon} F_{y'}(t)\eta'(t)\right] dt,\\ &= \int_{t_0}^{t_f} \left[F_{yy}(t)\eta^2(t) + 2F_{yy'}(t)\eta(t)\eta'(t) + F_{y'y'}(t)(\eta'(t))^2\right] dt.\end{aligned}$$

The integrand is a quadratic form[78]

$$\begin{bmatrix}\eta(t) & \eta'(t)\end{bmatrix}\begin{bmatrix}F_{yy} & F_{yy'}\\ F_{y'y} & F_{y'y'}\end{bmatrix}\begin{bmatrix}\eta(t)\\ \eta'(t)\end{bmatrix}.$$

[78] for each value of t, we have a different quadratic form!

If it is positive definite or negative definite for all $t \in [t_0, t_f]$, we have $\frac{\partial^2 V}{\partial \varepsilon^2} > 0$ or < 0.

We can make this more formal,

Theorem 17. *If $F[t, y, y']$ is concave (convex) in (y, y') then the Euler-Lagrange equation is sufficient for a global maximum (minimum) of $V[y(t)]$.*

Proof. Take the feasible path $y(t) = y^*(t) + \varepsilon\eta(t)$. Then, If F is concave in (y, y') then for all $(t, y, y'), (t, y^*, y'^*)$.

$$\begin{aligned}F(t, y, y') - F(t, y^*, y'^*) &\le F_y(t, y^*, y'^*)(y - y^*) + F_{y'}[t, y^*, y'^*](y' - y'^*),\\ &= \varepsilon\left(F_y(t, y^*, y'^*)\eta(t) + F_{y'}[t, y^*, y'^*]\eta'(t)\right).\end{aligned}$$

Integrating both sides from t_0 to t_f gives,

$$\begin{aligned}V[y(t)] - V[y^*(t)] &\le \varepsilon \int_{t_0}^{t_f} \left[F_y[t, y^*, y'^*]\eta(t) + F_{y'}[t, y^*, y'^*]\dot{\eta}(t)\right] dt,\\ &= \varepsilon \int_{t_0}^{t_f} \left[F_y[t, y^*, y'^*] - \frac{d}{dt} F_{y'}[t, y^*, y'^*]\right] \eta(t) dt,\\ &= 0\end{aligned}$$

□

The concavity (or convexity) of $F[t, y, \dot{y}]$ in (y, y') can be checked by considering the quadratic form $Q = F_{yy} dy^2 + 2F_{yy'} dy dy' + F_{y'y'} d(y')^2$. The function F is concave in (y, y') if Q is negative semi-definite everywhere.[79]

If the endpoints aren't fixed, this condition is, in general, not applicable. However, if the terminal line is vertical (truncated or not), the equation of Euler-Lagrange together with the appropriate transversality conditions are sufficient (otherwise, condition $[F_{y'}(y - y^*)]_{t=t_f} \le 0$ must be added.

[79] As stated above if the matrix

$$\begin{bmatrix}F_{yy} & F_{yy'}\\ F_{y'y} & F_{y'y'}\end{bmatrix}$$

is negative semi-definite for all t.

Consider an example. Let $V[y(t)] = \int_0^{t_f} \sqrt{1 + (y')^2} dt$ with $y(0) = A$ and $y(t_f) = Z$. Now, $F_{y,y}$ and $F_{yy'} = 0$ so,

$$\begin{aligned}F_{y'} &= \frac{y'}{\sqrt{1 + (y')^2}},\\ F_{y'y'} &= \frac{1}{(1 + (y')^2)\sqrt{1 + (y')^2}} > 0.\end{aligned}$$

This shows that F is convex so the extremal is a unique minimum.

As a second example consider $F(t) = 4y^2 + 4yy' + (y')^2$. We have,

$$\begin{aligned} F_y &= 8y + 4\dot{y}, \\ F_{y'} &= 4y + 2y', \\ F_{yy} &= 8, F_{yy'} = 4, F_{y'y'} = 2. \end{aligned}$$

So,

$$\begin{bmatrix} F_{yy} & F_{yy'} \\ F_{y'y} & F_{y'y'} \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}.$$

The eigenvalues are $\lambda = 0$ and $\lambda = 10$. The corresponding quadratic form is positive semi-definite everywhere, so F is convex.

THE SUFFICIENT CONDITION that F is concave or convex everywhere in (y, y') is rather strong. If this condition is not met, there is also a necessary second order condition, the so called **Legendre condition**. It is much weaker but not sufficient.

If $V[y(t)]$ reaches a maximum at $y^*(t)$. Then

$$F_{y'y'}[t, y^*(t), y'^*(t)] \leq 0.$$

IN MANY ECONOMIC models, there is no final period t_f, and one considers settings with $t_f = \infty$. These are the so called **infinite horizon problems**. In order to solve such problems, the idea is to extend the optimization problem on $[0, t_f] \to [0, \infty)$. As such, the value function becomes $\int_{t_0}^{\infty} F[t, y, y']dt$.

A crucial problem is the convergence of the value $\int_{t_0}^{\infty} F[t, y, y']dt$. This integral could either have a finite or infinite value. If $F[t, y, y']$ is everywhere finite and reaches 0 at some $t_f \in \mathbb{R}$ and remains equal to 0 after t_f, then the integral converges.

Also, if $F[t, y, y']$ can be written as $G[t, y, y']e^{-\rho t}$ with $\rho > 0$ and if G is bounded, then the integral also converges.[80]

[80] As an example, consider the utility maximization problem with objective function $\int_0^{\infty} e^{-\rho t} u(c(t))dt$. The problem here is to show that $u(c)$ is bounded.

$$\begin{aligned} \int_{t_0}^{\infty} G[t, y, y']e^{-\rho t} &\leq \int_{t_0}^{\infty} \bar{G}e^{-\rho t}, \\ &= \bar{G} \lim_{b \to \infty} [e^{-\rho t}/(-\rho)]_0^b, \\ &= \bar{G}(0 + 1/\rho), \\ &= \bar{G}/\rho. \end{aligned}$$

If F is continuous and non-negative then $\lim_{t \to \infty} F[t, y, y'] = 0$ is a necessary but not sufficient condition for convergence. If F is not continuous or not non-negative, above limit is not even necessary.

IN AN INFINITE horizon problem, there is obviously no fixed terminal time.

- If we take the transversality condition for t_f free, then the infinite horizon problem gives

$$\lim_{t\to\infty}[F - y'F_{y'}]_t = 0.$$

- If the asymptotically terminal value is fixed[81] it is obvious that $\lim_{t\to\infty} y(t) = y_\infty$ should be added as additional constraint.
- If the terminal value y_∞ is free, the following condition must be imposed $\lim_{t\to\infty}[F_{y'}]_t = 0$.
- If the free terminal state is subject to a lower bound $y(\infty) \geq y_{\text{min}}$ one first tries to solve $\lim_{t\to\infty} F_{y'}(t) = 0$. If the resulting $y(\infty) \geq y_{\text{min}}$ then everything is ok. Else, y_{min} is taken as the terminal value.

[81] Let us denote this by y_∞.

In a finite horizon problem, if $F[t, y, y']$ is concave in (y, y'), then the Euler-Lagrange equation is sufficient for a maximum of $V[y(t)]$ with fixed initial and terminal points. The latter condition remains sufficient if t_f is given, but $y(t_f)$ is not as long as $[F_{y'}(y - y^*)]_{t=t_f} \leq 0$

In the infinite horizon framework, this extra condition becomes

$$\lim_{t\to\infty}[F_{y'}(t)(y(t) - y^*(t))] \leq 0.$$

The Ramsey problem

Consider an economy with capital stock $K(t)$. The level of consumption is denoted by $C(t)$ and output is given by the production function $Y(t) = F(K(t))$. We assume that $F' > 0$ and $F'' \leq 0$.[82] Investment, i.e., growth of capital is determined by the amount of output that is left after consumption.

[82] The production function is increasing and concave.

$$K'(t) = Y(t) - C(t) = F(K(t)) - C(t).$$

Assume a representative consumer with instantaneous utility function $U(C(t))$, with a life span of T years. The consumer has a discount rate equal to r. The consumer then solves

$$\max_C \int_0^T e^{-rt} U(C(t)) dt,$$
$$\text{subject to } K'(t) = F(K(t)) - C(t); K(0) = \bar{K}; K(T) = 0.$$

Substituting the constraint into the objective gives,

$$\max_K \int_0^T e^{-rt} U(K' - F(K)) dt \text{ subject to } K(0) = \bar{K}; K(T) = 0.$$

The Euler-Lagrange condition is given by,

$$\begin{aligned}
&F_K - \frac{d}{dt}F_{K'} = 0,\\
\leftrightarrow &-e^{-rt}U'(C)F'(K) - \frac{d}{dt}\left[e^{-rt}U'(C)\right] = 0,\\
\leftrightarrow &-U'(C)F'(K) + rU'(C) - U''(C)C' = 0,\\
\leftrightarrow &-\frac{U''(C)}{U'(C)}C' = F'(K) - r.
\end{aligned}$$

Define the elasticity of marginal utility as $\eta(C) = -\frac{\partial \ln(U'(C))}{\partial \ln(C)} = -\frac{U''(C)}{U'(C)}C$.[83] Then,

$$\eta(C)\frac{C'}{C} = F'(K) - r.$$

The left hand side is the proportional growth of consumption multiplied by the elasticity of marginal utility. Or the elasticity of intertemporal rate of substitution.

Let us consider the special case of a CRRA utiltiy function $U(C) = \frac{C^{1-\eta}}{1-\eta}$ with $\eta \in [0,1]$.[84] Then $\eta(C) = \eta$ and the equation gives,

$$\begin{aligned}
\eta\frac{C'}{C} &= F'(K) - r,\\
\frac{C'}{C} &= \frac{F'(K) - r}{\eta}.
\end{aligned}$$

So the growth rate in consumption is equal to the product of the 'intertemporal elasticity of substitution' $\frac{1}{\eta}$ and the difference between the marginal product of capital $F'(K)$ and the discount rate r.

[83] This is equal to the rate of relative risk aversion. It measures the curvature of the utility function.

[84] CRRA stands for constant relative risk aversion.

Exercises

SOLVE THE FOLLOWING maximization problem using the Euler equation:

- $V[y(t)] = \int_0^1 (t + y'^2)dt$ with $y(0) = 0$ and $y(1) = 2$. (sol: $y = 2t$)
- $V[y(t)] = \int_0^2 (7y'^3)dt$ with $y(0) = 9$ and $y(2) = 11$. (sol: $y = t + 9$)
- $V[y(t)] = \int_0^1 (y + yy' + y' + \frac{1}{2}y'^2)dt$ with $y(0) = 2$ and $y(1) = 5$. (sol $y = t^2/2 + 5t/2 + 2$.)
- $V[y(t)] = \int_0^{\pi/2} (y^2 - y'^2)dt$ with $y(0) = 0$ and $y(\pi/2) = 1$. (sol $y = \sin t$)

- $V[y(t)] = \int_0^1 (1 + y''^2)dt$ with $y(0) = 0$ and $y'(0) = 1$, $y(1) = 1$ and $y'(1) = 1$.
- $V[y(t)] = \int_0^T (t^2 + y'^2)dt$ with $y(0) = 4$, $y(T)$ free and $T = 2$. (sol: $y = 4$)
- $V[y(t)] = \int_0^T (t^2 + y'^2)dt$ with $y(0) = 4$, $y(T) = 5$, T free. (sol: $y = t + 4$, $T = 1$)

Is F concave or convex in (y, y')?

- $V[y(t)] = \int_0^{t_f} (t + (y')^2)dt$. (Ans: convex)
- $V[y(t)] = \int_0^{t_f} (y + yy' + y' + (y')^2/2)dt$ with $y(0) = 2$ and $y(1) = 5$. (Ans: neither convex nor concave)
- $V[y(t)] = \int_0^{t_f} (y^2 + 4yy' + 4(y')^2)dt$ with $y(0) = 2e^{1/2}$ and $y(1) = 1 + e$. (Ans: convex).

Optimal control theory

The calculus of variations is the classical method to approach dynamic optimization problems but is not frequently used anymore. First, it assumes differentiability and even more of the functions present in the problem. Also, only interior solutions are considered.

The theory known as **optimal control** allows, among other things, to take into account corner solutions and solutions that give functions which are not everywhere differentiable.

In the calculus of variation framework, the optimal path $y^*(t)$ of the state variable $y(t)$ is the goal. In contrast, in the optimal control framework, one first concentrates on the optimal path(s) of one (or several) control variables, denoted by $u(t)$. Once $u^*(t)$ is determined, the optimal path $y^*(t)$ is obtained.[85]

[85] In fact both optimal paths are generally obtained simultaneously.

What makes a variable be called a control variable is first that its path may be chosen,[86] and second that it drives the state variable in such a way to optimize the objective functional.

[86] For example by a social planner

Let us have a look at a problem involving only one control variable $u(t)$ and one state variable $y(t)$. Here $u(t)$ is a policy instrument which influences or drives the state variable $y(t)$.[87] To each path of $u(t)$ corresponds one unique path of $y(t)$ and what is sought is the optimal path of $u^*(t)$ along with the corresponding path of $y^*(t)$ that optimizes the objective functional.

[87] For example, investment determines the capital stock.

1. The control path $u(t)$ does not necessarily have to be continuous in order for it to be admissible. Only piecewise continuity is required.

2. The path of the state variable $y(t)$ is required to be C^0 on $[t_0, t_f]$ but not necessarily differentiable on $[t_0, t_f]$,[88] only piecewise differentiability is required. Any finite number of points of non-differentiability is allowed for $y(t)$.

[88] C^0 means that it is continuous.

3. Additionally, optimal control theory allows to take into account a constraint on control variables $u(t)$ such as imposing that it belongs to a bounded subclass of functions.

4. The simplest problem of optimal control leaves the terminal point $y(t_f)$ free.[89]

[89] This is for theoretical reasons, the freedom in the terminal value makes the problem easier to handle.

Consider the problem

$$\max_{u(t)} V[u(t), y(t)] = \int_{t_0}^{t_f} F[t, y(t), u(t)]dt,$$
$$\text{s.t. } y'(t) = f(t, y(t), u(t)),$$
$$u(t) \in \mathcal{U}(t), y(t) \in \mathcal{Y}(t), y(t_0) = y_0.$$

where $\mathcal{U}(t), \mathcal{Y}(t) \subseteq \mathbb{R}$ and $F, f : \mathbb{R}^3 \to \mathbb{R}$. The variable y is called the **state variable**. Its value is governed by its initial value y_0 and the differential equation

$$y'(t) = f(t, y(t), u(t)).$$

This equation is called the state equation or **equation of motion**.[90]

[90] In the special case where $y'(t) = u(t)$, this problem reduces to the problem we encountered in the calculus of variations.

The function $V[u(t), y(t)]$ denotes the value of the objective function when controls are given by $u(t)$ and the behaviour of the state variable is summarized by $y(t)$.

A PAIR OF functions $(y(t), u(t))$ such that

$$u(t) \in \mathcal{U}(t),$$
$$y(t) \in \mathcal{Y}(t),$$
$$y(t_0) = y_0,$$
$$y' = f(t, y, u).$$

is called an **admissible pair**. We assume that $V[u(t), y(t)]$ is finite for all admissible pairs.

For simplicity, we also assume that both F and f are continuously differentiable in t, y and u. The problem in characterizing the optimal solution lies in two things.

1. We are looking for a function $y : [t_0, t_f] \to \mathbb{R}$ rather than a single value.

2. The constraint is a differential equation rather than a set of equalities or inequalities.

We can make our lives a lot easier by assuming that the solutions $(u^*(t), y^*(t))$ lie in the interior of the sets $\mathcal{U}(t)$ and $\mathcal{Y}(t)$ and that u^* is continuous. Given that $(u^*(t), y^*(t))$ is optimal, it must be that,

$$V(u^*(t), y^*(t)) \geq V(u(t), y(t)),$$

for all admissible pairs $(u(t), y(t))$ in a local neighbourhood of $(u^*(t), y^*(t))$.

AS BEFORE, WE look at small variations of the optimal solution of the control variable,

$$
\begin{aligned}
u(t, \varepsilon) &= u^*(t) + \varepsilon \eta(t), \\
\eta(t_0) &= 0.
\end{aligned}
$$

Let us define $y(t, \varepsilon)$ as the unique path of the state variable that is determined by path of the control $u(t, \varepsilon)$, i.e.,

$$
\begin{aligned}
y_t(t, \varepsilon) &= f(t, y(t, \varepsilon), u(t, \varepsilon)), \\
y(t_0, \varepsilon) &= y_0.
\end{aligned}
$$

Define,

$$V(\varepsilon) = \int_{t_0}^{t_f} F[t, y(t, \varepsilon), u(t, \varepsilon)] dt.$$

Then we have,

$$V(\varepsilon) \leq V(0),$$

for all ε in a neighbourhood around zero. Given that $(y(t, \varepsilon), u(t, \varepsilon))$ satisfies the equation of motion, we have that,

$$\int_{t_0}^{t_f} \lambda(t)[f(t, y(t, \varepsilon), u(t, \varepsilon) - y_t(t, \varepsilon)] dt = 0.$$

for all functions $\lambda : [t_0, t_f] \to \mathbb{R}$. We take $\lambda(t)$ to be C^1. The function $\lambda(t)$ is called the **costate variable** and its interpretation is similar to the Lagrange multiplier found in static optimization models.[91] Similar to static optimization models, only suitable $\lambda(t)$ paths will play the role of costate variable. We have that,

[91] It is the marginal value of relaxing the equation of motion.

$$V[\varepsilon] = \int_{t_0}^{t_f} \left(F(t)t, y(t, \varepsilon), u(t, \varepsilon) + \lambda(t)(f(t, y(t, \varepsilon), u(t, \varepsilon)) - y_t(t, \varepsilon)) \right) dt,$$

It will be convenient to rewrite the term $\int_{t_0}^{t_f} \lambda(t) y_t(t, \varepsilon) dt$ using integration by parts,

$$\int_{t_0}^{t_f} \lambda(t) y_t(t, \varepsilon) dt = \lambda(t_f) y(t_f, \varepsilon) - \lambda(t_0) y_0 - \int_{t_0}^{t_f} y(t, \varepsilon) \lambda'(t) dt.$$

Substituting back,

$$
\begin{aligned}
V[\varepsilon] = &\int_{t_0}^{t_f} \left(F[t, y(t, \varepsilon), u(t, \varepsilon) + \lambda(t) f(t, y(t, \varepsilon), u(t, \varepsilon) + \lambda'(t) y(t, \varepsilon) \right) dt, \\
&- \lambda(t_f) y(t_f, \varepsilon) + \lambda(t_0) y_0.
\end{aligned}
$$

Differentiating with respect to ε, making use of Leibnitz' rule,

$$\begin{aligned} V'[\varepsilon] = &\int_{t_0}^{t_f} \left(F_y[t, y(t,\varepsilon), u(t,\varepsilon)] + \lambda(t) f_y(t, y(t,\varepsilon), u(t,\varepsilon)) + \lambda'(t)\right) y_\varepsilon(t,\varepsilon) dt, \\ &+ \int_{t_0}^{t_f} \left(F_u[t, y(t,\varepsilon), u(t,\varepsilon)] + \lambda(t) f_u(t, y(t,\varepsilon), u(t,\varepsilon))\right) \eta(t) dt, \\ &- \lambda(t_f) y_\varepsilon(t_f, \varepsilon). \end{aligned}$$

Evaluating at $\varepsilon = 0$ gives,

$$\begin{aligned} 0 = &\int_{t_0}^{t_f} \left(F_y(t) + \lambda(t) f_y(t) + \lambda'(t)\right) y_\varepsilon(t,0) dt, \\ &+ \int_{t_0}^{t_f} \left(F_u(t) + \lambda(t) f_u(t)\right) \eta(t) dt, \\ &- \lambda(t_f) y_\varepsilon(t_f, 0). \end{aligned}$$

This has to hold for all $\eta(t)$ and all functions $\lambda(t)$. Consider the function $\lambda(t)$ obtained by,

$$\lambda'(t) = -F_y(t) - \lambda(t) f_y(t),$$

with boundary condition $\lambda(t_f) = 0$. This gives,

$$0 = \int_{t_0}^{t_f} \left(F_u(t) + \lambda(t) f_u(t)\right) \eta(t) dt.$$

As this has to hold for all $\eta(t)$, we obtain, that for all $t \in [t_0, t_f]$,

$$F_u(t) + \lambda(t) f_u(t) = 0,$$

The condition $\lambda(t_f) = 0$ is called the transversality condition.[92] Summarizing, we have the following set of conditions,

$$\begin{aligned} &F_u(t) + \lambda(t) f_u(t) = 0, \\ &\lambda'(t) = -F_y(t) - \lambda(t) f_y(t), \\ &\lambda(t_f) = 0. \end{aligned}$$

[92] Intuitively, this condition captures the fact that after the planning horizon, there is not additional value to changing $y'(t_f)$.

A MORE ECONOMICAL way of expressing the optimality conditions is by introducing the **Hamiltonian**,

$$\begin{aligned} H(t, y(t), u(t), \lambda(t)) &= F[t, y(t), u(t)] + \lambda(t) f(t, y(t), u(t)), \\ &= F(t) + \lambda(t) f(t). \end{aligned}$$

We can summarize the conditions found above by using the Hamiltonian. Necessary conditions for $u^*(t)$ and $y^*(t)$ to be optimal, i.e. **the**

maximum principle, are that for all $t \in [0,T]$,

$$H[t, y^*, u^*, \lambda^*] = \max_u H[t, y^*, u, \lambda^*],$$
$$y'^* = H_\lambda(t, y^*, u^*, \lambda^*) = f(t, y^*, u^*),$$
$$\lambda'^* = -H_y(t, y^*, u^*, \lambda^*) = -F_y(t, y^*, u^*) - \lambda(t) f_y(t, y^*, u^*),$$
$$\lambda^*(t_f) = 0.$$

The first condition becomes

$$H_u(t) = F_u(t) + \lambda(t) f(t) = 0,$$

if H is differentiable in u and the solution is interior.[93] Also, the second condition simply gives the equation of motion $y' = f(t, y, u)$. The third condition reads,

[93] On the other hand, if H is (for example) linear in u, the maximum will be reached at a corner.

$$\lambda'(t) = -H_y(t) = -F_y(t) - \lambda(t) f_y(t).$$

Let us maximize $V[y(t), u(t)] = \int_0^{t_f} -(1+u^2)^{1/2} dt$ subject to $y' = u$, with $y(0) = A, y(t_f)$ free. The Hamiltonian is given by,

$$H = -(1+u^2)^{1/2} + \lambda u.$$

Then

$$\frac{\partial H}{\partial u} = \frac{-u}{\sqrt{1+u^2}} + \lambda = 0,$$
$$\lambda' = -\frac{\partial H}{\partial y} = 0,$$
$$y' = u,$$
$$\lambda(t_f) = 0.$$

The second condition gives $\lambda(t) = K$, a constant. The transversality condition, on the other hand gives $K = 0$. As such, substituting $\lambda = 0$ into the the first condition then gives $u(t) = 0$ for all t. But then the equation of motion gives $y' = 0$ which means that $y^*(t) = A$, a constant.

As a second example consider maximizing $V[y(t), u(t)] = \int_0^2 (2y - 3u) dt$ subject to $y' = y + u$. with $y(0) = 4, y(2)$ free and $u(t) \in [0, 1]$.

The Hamiltonian is $H = (2y - 3u) + \lambda(y + u)$. The Hamiltonian is linear in u with derivative $-3 + \lambda$. So if $\lambda > 3$, $u^* = 2$ if $\lambda < 3$, $u^* = 0$.

The other condition gives

$$\lambda' = -2 - \lambda.$$

This is a linear first order equation with general solution $\lambda(t) = Ae^{-t} - 2$. The condition $\lambda(t_f) = 0$ gives $0 = Ke^{-2} - 2$ or $K = 2e^2$. So $\lambda^*(t) = 2e^{2-t} - 2$.

On the interval $[0, 2]$, $\lambda^*(t)$ decreases from $2e^2 - 2 > 3$ to $0 < 3$. $\lambda(t) = 3$ if $e^{2-t} = 2.5$, so $t = 2 - \ln(2.5)$. Then $u^*(t) = 2$ in $[0, 2 - \ln(2.5)]$ and $u^*(t) = 0$ for $t \in [2 - \ln(2.5), 2]$.

This is an example of a Bang-Bang control. The equation of motion is $y' = y + u$ so in phase one this is $y(t) = We^t - 2$. In phase 2, we have $y' = y$, then the general solution is Le^t. The initial condition fixes $W = 6$. Continuity of y^* requires that $6e^{2-\ln(2.5)} - 2 = Le^{2-\ln(2.5)}$ so $L = 6 - 2e^{-2+\ln(2.5)}$.

For other terminal conditions, we get other transversality conditions.

- Fixed terminal point
  In case the terminal point $y(t_f) = y_f$ with y_f is fixed. The perturbation $\eta(t)$ can not be arbitrary as $y' = f(t, y, u)$ must restrict $u(t)$ such that $y^*(t_f) = y_f$. It can be shown that the maximum principle remains valid if the terminal condition $\lambda(t_f) = 0$ is replaced by the terminal condition $y(t_f) = y_f$.

- Horizontal terminal line ($y(t_f)$ given, t_f free)
  In this setting, the transversality condition is given by $[H]_{t_f} = 0$.

- Terminal curve ($y_f = \phi(t_f), \phi$ given)
  It can be proven in this case, that the transversality condition is now $[H - \lambda\phi']_{t=t_f} = 0$.

- Truncated vertical terminal line (t_f given, $y(t_f) \geq y_{\min}$)
  As before, either $y^*(t_f) > y_{\min}$ or $y^*(t_f) = y_{\min}$ A simple reasoning yields the following complementary slackness condition transversality condition,

  $$[(y - y_{\min})\lambda]_{t_f} = 0.$$

  From a practical point of view, let $\lambda(t_f) = 0$ determine $y(t_f)$. If the latter gives $y(t_f) \geq y_{\min}$, then everything is ok. Else, set $y(t_f) = y_{\min}$ and consider a problem with a fixed terminal point.

- Truncated horizontal terminal line (y_{t_f} given, $t_f \leq t_{\max}$)
  As in the previous case, we necessarily have,

$$[(t - t_{\max})H]_{t_f} = 0.$$

IF THE HAMILTONIAN $H[t, y, u, \lambda]$ is differentiated with respect to t, we have

$$\frac{d}{dt}[H] = H_t + H_y y' + H_u u' + H_\lambda \lambda'.$$

The necessary condition $H_u = 0$, and condition $H_y = -\lambda', H_\lambda = y'$ imply that,

$$\frac{d}{dt}[H] = H_t.$$

If the problem is **autonomous**, i.e. if t is not present as a separate argument in H, then $H_t = 0$, which means that the value of the Hamiltonian at the optimal solution $H(y^*(t), u^*(t))$ is a constant.

THE SIMPLEST PROBLEM of the calculus of variations is

$$\begin{aligned} &\max V[y(t)] = \int_{t_0}^{t_f} F[t, y(t), y'(t)]dt, \\ &\text{s.t. } y(t_0) = y_0; y(t_f) = y_f. \end{aligned}$$

If $y'(t)$ is substituted by $u(t)$, we get,

$$\begin{aligned} &\max V[y(t), u(t)] = \int_{T_0}^{t_f} F[t, y(t), u(t)]dt, \\ &\text{s.t. } y' = u, \\ &\quad y(t_0) = y_0, y(t_f) = y_f. \end{aligned}$$

which is a simple optimal control problem with Hamiltonian $H = F[t, y(t), u(t)] + \lambda u$. The maximum principle gives,

$$\begin{aligned} H_u &= 0 \to F_{y'} + \lambda = 0, \\ \lambda' &= -\frac{\partial H}{\partial y} = -F_y, \\ y' &= H_\lambda = u. \end{aligned}$$

So we have $\lambda = -F_{y'}$ and $\lambda' = -F_y$. This gives $\frac{d}{dt}[-F_{y'}] = -F_y$ which is the Euler-Lagrange condition. The transverality condition $\lambda(t_f) = 0$ is equal to the condition $[F_u]_{t_f} = [F_{y'}]_{t_f} = 0$ which is the transversality condition for a free terminal value.

Consider the problem of maximizing

$$\begin{aligned} &\max V[y(t)] = \int_0^1 (-u^2)dt, \\ &\text{s.t. } y' = y + u, \\ &\quad y(0) = 1, \\ &\quad y(1) = 0. \end{aligned}$$

We have $H = (-u^2) + \lambda(y + u)$ nonlinear in u. So,

$$\begin{aligned} &\frac{\partial H}{\partial u} = 0 \to -2u + \lambda = 0 \\ &\lambda' = -\frac{\partial H}{\partial y} \to \lambda' = -\lambda, \\ &y' = \frac{\partial H}{\partial \lambda} \to y' = y + u. \end{aligned}$$

This gives, $u = \lambda/2$, $\lambda' + \lambda = 0$ or $\lambda(t) = Ke^{-t}$ and $y' = y + Ke^{-t}$. Solving this gives, $y(t) = We^t - (K/4)e^{-t}$.

The initial condition gives $1 = W - (K/4)$ and $0 = We - K/(4e)$. Then $K = 4e^2/(1 - e^2)$ and $W = 1/(1 - e^2)$.

As another example, consider

$$\begin{aligned} &\max V = \int_0^1 (-1)dt, \\ &\text{s.t. } y' = y + u, \\ &\quad y(0) = 5, \\ &\quad y(t_f) = 11, \\ &\quad T \text{ is free }, \\ &\quad u(t) \in \mathcal{U} = [-1, 1]. \end{aligned}$$

The Hamiltonian is $H = -1 + \lambda(y + u)$. We see that H is linear in u so it reaches a maximum at one of the extremes, -1 or 1. We have that $u^*(t) = 1$ if $\lambda > 0$ and $u^*(t) = -1$ if $\lambda < 0$.

Also, $\lambda' = -\frac{\partial H}{\partial y} = -\lambda$. So $\lambda(T) = Ke^{-t}$.

The sign of λ is that of K for all $t > 0$. Consequently, a bang-bang phenomenon occurs even though $\mathcal{U} = [-1, 1]$. The transversality condition is $H(t_f) = 0$ which requires $-1 + Ke^{-t_f}(11 + u^*) = 0$. As $11 + u^* > 0$, $K > 0$. Then $\lambda(t) > 0$ for all t and $u^*(t) = 1$.

In addition $y' = \frac{\partial H}{\partial \lambda} = y + u$, so $y' = y + 1$ and $y = \bar{K}e^t - 1$. From $y(0) = 5, \bar{K} = 6$ and $y^*(t) = 6e^t - 1$.

Going back to the transversaility condition $H(t_f) = 0$ we get $-1 + Ke^{-t_f}(11 + 1) = 0$ so $K = e^{t_f}/12$ and we know $11 = 6e^{t_f} - 1$ or $e^{t_f} = 2$. From this $K = 1/6$. Conclude that $\lambda^*(t) = e^{-}1/6$. Finally $t_f = \ln(2)$.

If $y(t_f)$ is given but not the terminal time t_f, we have the condition

$$\begin{aligned} &H(t_f) = 0, \\ \leftrightarrow &[F + \lambda u]_{t=t_f} = 0, \\ \leftrightarrow &[F - y' F_{y'}]_{t=t_f} = 0. \end{aligned}$$

This is the transversality condition for the calculus of variations.

The current value Hamiltonian

In many economic models, the integrand function $F[t, y, u]$ is of the form $G[t, y, u]e^{-\rho t}$. In this case, we can write the problem as,

$$\max \int_{t_0}^{t_f} e^{-\rho t} G[t, y, u] dt,$$
$$\text{s.t. } y' = f(t, y, u).$$

The Hamiltonian is

$$H = e^{-\rho t} G[t, y, u] + \lambda f(t, y, u).$$

In order to analyse this problem, one often makes use of another Hamiltonian, called the **current value hamiltonian**, which is obtained by multiplying the Hamiltonian by $e^{\rho t}$.

$$C = He^{\rho t} = G[t, y, u] + \lambda(t)e^{\rho t} f(t, y, u) = G[t, y, u] + m(t) f(t, y, u).$$

Where we introduced the multiplier $m(t) = \lambda(t)e^{\rho t}$.

The conditions are somewhat different,

1. u maximizing H becomes that u maximizes C.
2. $y' = H_\lambda$ becomes $y' = C_m$.
3. $\lambda' = -H_y$ becomes $m' - \rho m = -C_y$.
4. the transversality condition $\lambda(t_f) = 0$ becomes $m(t_f)e^{-\rho t_f} = 0$.
5. the transversality condition $[H]_{t_f} = 0$ becomes $[C]_{t_f} e^{-\rho t_f} = 0$.

In order to get the intuition, observe,

$$\begin{aligned} &H_u = e^{-\rho t} C_u = 0 \leftrightarrow C_u = 0 \\ &y' - H_\lambda = y' - C_m = 0, \\ &\lambda' = -H_y \leftrightarrow m' e^{-\rho t} - \rho m e^{-\rho t} = -e^{-\rho t} C_y \leftrightarrow m' - \rho m = -C_y. \end{aligned}$$

The current value Hamiltonian is frequently used in economics. However, care must be taken as the interpretation of the multiplier changes a bit. The multiplier $\lambda(t)$ measures the shadow price of $y'(t)$ in period t_0 terms (you're optimizing the discounted sum so everything gets discounted to the present). This is also, why H if called the present value Hamiltonian. On the other hand, the multiplier $m(t)$ for the current value Hamiltonian gives the shadow price of $y'(t)$ measured in period t units.

Consider the infinite horizon problem. The transversality condition states that $\lim_{t\to\infty} \lambda(t) = 0$ meaning that the present value shadow price of an increase in $y'(t)$ goes to zero. The corresponding transversality condition for the current value Hamiltonian is that $\lim_{t\to\infty} m(t)e^{-\rho t} = 0$. This condition allows $m(t)$ to go to infinity as $t \to \infty$. However, the increase in $m(t)$ should be slower than the exponential growth of $e^{\rho t}$.

Sufficient conditions

FOR THE STANDARD problem, the necessary conditions of the maximum principle are sufficient for a global maximum of V if

- F and f are differentiable and concave in (y, u).
- $\lambda^*(t) \geq 0$ for all $t \in [t_0, t_F]$ if f is non-linear in y or in u.

These are called **Mangassarian's conditions**. To see that they are sufficient, assume that F and f are concave in (y, u). Then, we have for any (t, y, u) and (t, y^*, u^*) in their domains,

$$F[t,y,u] - F[t,y^*,u^*] \leq F_y[t,y^*,u^*](y-y^*) + F_u[t,y^*,u^*](u-u^*),$$
$$f(t,y,u) - f(t,y^*,u^*) \leq f_y(t,y^*,u^*)(y-y^*) + f_u(t,y^*,u^*)(u-u^*).$$

Then,[94]

[94] The next to last line uses integration by parts. The last line follows from the transversality condition $\lambda(t_f) = 0$ and the initial condition $y(t_0) = y^*(t_0) = y_0$.

$$
\begin{aligned}
V[y,u] - V[y^*,u^*] &\leq \int_{t_0}^{t_f} [F_y[t,y^*,u^*](y-y^*) + F_u[t,y^*,u^*](u-u^*)]dt, \\
&= \int_{t_0}^{t_f} \left[\left(-\lambda'^*(t) - \lambda^*(t) f_y(t,y^*,u^*)\right)(y-y^*) - \lambda^*(t) f_u(t,y^*,u^*)(u-u^*) \right] dt, \\
&= \int_{t_0}^{t_f} -\lambda'^*(t)(y-y^*)dt, \\
&+ \int_{t_0}^{t_f} \lambda^* \left[-f_y(t,y^*,u^*)(y-y^*) - f_u(t,y^*,u^*)(u-u^*) \right] dt, \\
&\leq \int_{t_0}^{t_f} -\lambda'^*(t)(y-y^*)dt, \\
&+ \int_{t_0}^{t_f} \lambda^* \left[f(t,y^*,u^*) - f(t,y,u) \right] dt, \\
&= \int_{t_0}^{t_f} -\lambda'^*(t)(y-y^*)dt, \\
&+ \int_{t_0}^{t_f} \lambda^* \left[y'^* - y' \right] dt, \\
&= -\lambda(t_f)(y(t_f) - y^*(t_f)) + \lambda(t_0)(y(t_0) - y^*(t_0)) + \int_{t_0}^{t_f} \lambda^*(y' - y'^*)dt, \\
&+ \int_{t_0}^{t_f} \lambda^* \left[y'^* - y' \right] dt, \\
&= 0.
\end{aligned}
$$

It is easy to show that the theorem remains true if $y(t_f) = y_f$ is given and even if the terminal line is (vertical) or truncated.

THE SUFFICIENT CONDITION of **Arrow** is weaker than Mangasarian's. Define the Maximized Hamiltonian $M(t, y(t), \lambda(t))$ as follows

$$
M(t,y(t),\lambda(t)) = \max_{u(t)} H(t,y(t),u(t),\lambda(t)).
$$

The sufficient condition of Arrow requires that M is concave in y.

Consider any other feasible path $u(t)$ and $y(t)$. Then,[95]

[95] The last inequality follows from concavity of M in y (for fixed λ).

$$
\begin{aligned}
F(t,y,u) + \lambda^* f(t,y,u) &\leq M(t,y,\lambda^*), \\
&\leq M(t,y^*,\lambda^*) + M_y(t,y^*(t),\lambda(t)^*)(y(t) - y^*(t)).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&F(t,y,u) + \lambda^* f(t,y,u), \\
&\leq F(t,y^*,u^*) + \lambda^* f(t,y^*,u^*) + M_y(t,y^*,\lambda^*)(y-y^*).
\end{aligned}
$$

From the Envelope theorem and the first order conditions,

$$
M_y(t,y^*,\lambda^*) = H_y(t,y^*,u^*,\lambda^*) = -\lambda^{*\prime},
$$

So,

$$F(t,y,u) \leq F(t,y^*,u^*) + \lambda^*(y'^* - y') - \lambda^{*\prime}(y-y^*).$$

Taking the integral on both sides, gives,

$$\int_{t_0}^{t_f} F(t,y,u)dt \leq \int_0^{t_f} F(t,y^*,u^*)dt + \int_0^{t_f} \lambda^*(y'-y'^*)dt - \int_0^{t_f} \lambda^{*\prime}(y-y^*)dt.$$

Integrating by parts, and using the fact that $y(t_0) = y^*(t_0) = y_0$ and $\lambda^*(t_f) = 0$ gives,

$$\int_{t_0}^{t_f} F(t,y,u)dt \leq \int_{t_0}^{t_f} F(t,y^*,u^*)dt.$$

Summarizing, the **sufficient condition of Arrow** is that,

- M is concave in y for all $t \in [t_0, t_f]$ and λ fixed.
- As long as t_f is given, the theorem can be extended to terminal conditions y_{t_f} given and truncated terminal lines.
- The latter theorem can also be re-written in terms of the current Hamiltonian C.

Variations

FOR PROBLEMS OF many state and control variables, the problem can be formulated as follows,

$$\max V = \int_{t_0}^{t_f} F[t, y^1, y^2, \dots, y^m, u^1, u^2, \dots, u^m]dt,$$
$$\text{s.t. } y'^j = f(t, y^1, y^2, \dots, y^m, u^1, u^2, \dots, u^m), \ j = 1, \dots, m$$

with $y^j(t_0) = y_0^j$ and $y^j(t_f) = y_f^j$

Define $H = F[t, y^1, \dots, y^m, u^1, \dots, u^m] + \sum_j^n \lambda^j f^j(t, y^1, \dots, y^m, u^1, \dots, u^m)$. Then, the maximum principle becomes

$$(u^{*,1}, \dots, u^{*,m}) \in \arg\max_{u^1,\dots u^m} H,$$
$$y'^j = H_{\lambda^j},$$
$$\lambda^{j\prime} = -H_{y^j}.$$

FOR INFINITE PROBLEMS the transversality condition has to be adjusted. The general condition is given by,

$$\lim_{t\to\infty}[H] = 0.$$

Consider the problem,

$$\max V[y,u] = \int_0^{t_f} -(1+u^2)^{1/2}dt,$$
$$\text{s.t. } y' = u,$$
$$y(0) = y_0, y(t_f) \text{ free}.$$

We will check the sufficient conditions of Mangasarian and of Arrow.

For Mangasarian, $F = -(1+u^2)^{1/2}$ only depends on u. We have,

$$F_u = -\frac{u}{\sqrt{1+u^2}},$$
$$F_{uu} = \frac{-\sqrt{1+u^2} + \frac{u^2}{\sqrt{1+u^2}}}{(1+u^2)} = \frac{-1}{(1+u^2)^{2/3}}.$$

We see that $F_{uu} < 0$ so F is concave in u. As f is equal to u, it is linear in u (and in y), thus concave in u. As f is linear, there is not need for $\lambda(t) \geq 0$.

For Arrow, we have $u^* = \frac{\lambda}{(1-\lambda^2)^{1/2}}$.

$$M(t,y,\lambda) = F[t,y,u^*] + \lambda f(t,y,u^*),$$

is independent of y, as such, it is also concave in y.

For a second example consider

$$\max V[u] = \int_0^1 -u^2 dt,$$
$$\text{s.t. } y' = y + u,$$
$$y(0) = 1, y(1) = 0.$$

We have $F[t,y,u] = -u^2$, which is clearly concave in (y,u). Furthermore, $f(t,y,u) = y+u$, linear in y and u. As such, it is also concave in (y,u)

For Arrow, as $u^* = \lambda/2$, $M = -(u^*)^2 + \lambda(y+u^*) = -\lambda^2/4 + \lambda(y + \lambda/2)$ which is linear in y for given λ.

For a final example, consider

$$\max \int_0^{t_f} (-1)dt,$$
$$\text{s.t. } y' = 2u,$$
$$y(0) = 8, y(t_f) = 0, u(t) \in [-1,1].$$

We had $u^*(t) = -1$ so $M = (-1) + \lambda 2(-1)$ which is independent of y so concave in y.

Also $F = -1$ which is concave in (y,u) and $f = 2u$ which is concave in (y,u). But, t_f is free ...

If the terminal state y_∞ is fixed, we quite naturally have

$$\lim_{t\to\infty} y(t) = y_\infty.$$

If not, $\lim_{t\to\infty} \lambda(t) = 0$ is a necessary transversality condition. Some authors contest the validity of the latter transversality condition.

The maximum principle is sufficient for a global maximum if either H is concave in (y, u) or $M(t, y, \lambda)$ is concave in y for λ given and $\lim_{t\to\infty} \lambda(t)[y(t) - y^*(t)] \geq 0$.

Optimal growth model

Consider a representative consumer maximizing the present value of utility of consumption for society, and also accumulate a specified capital stock by the end of the horizon.

The stock of capital $K(t)$ is the only factor of production. Let $F(K)$ be the output. Assume $F(0) = 0, F'(K) > 0, F''(K) < 0$. This implies decreasing marginal productivity. Let $C(t)$ be the amount of output allocated to consumption and let $I(t) = F(K(t)) - C(t)$ be the amount invested. Let δ be the constant rate of depreciation of capital. Then the capital stock equation is given by,

$$K'(t) = I(t) - \delta K(t) = F(K(t)) - C(t) - \delta K(t).$$

Let $u(C)$ be societies level of utility. We assume $u(0) = 0, u(c) \geq 0, u'(C) > 0, u''(C) < 0$. Let ρ denote societies discount rate and T the planning horizon. Then the problem is,

$$\begin{aligned}
&\max \int_0^{t_f} u(C(t))dt, \\
&\text{s.t. } K' = F(K) - C - \delta K, \\
&\quad\;\; K(0) = K_0, K(t_f) = K_{t_f}.
\end{aligned}$$

The current value Hamiltonian is given by,

$$u(C) + m(F(K) - C - \delta K).$$

The maximal principle gives the conditions

$$\begin{aligned}
&u'(C) - m = 0, \\
&K' = f(K) - C - \delta K, \\
&m' - \rho m = -m(F'(K) - \delta).
\end{aligned}$$

Differentiating the first condition with respect to time gives,

$$u''(C)C' = m' = -m(F'(K) - \delta - \rho).$$

Using again the first condition gives,

$$\begin{aligned}
&u''(C)C' = -u'(C)(f'(K) + \delta - \rho), \\
\leftrightarrow &\frac{C'}{C} = \left(-\frac{u''(C)C}{u'(C)}\right)^{-1} (F'(K) - \delta - \rho).
\end{aligned}$$

This is the famous **Euler equation**. On the left hand side, you have the growth rate of consumption. The first term on the right is the inverse of the measure of relative risk aversion. This shows that the growth rate is higher the higher the marginal product of capital, the lower the depreciation rate and the lower the discount rate.

Now, assume there is a a new factor of production labour (which for simplicity we treat the same as the population), which is growing exponentially at a fixed rate $g > 0$. Let $L(t)$ denote the amount of labour at time t.

Let $F(K, L)$ be the production function which is assumed to be concave and homogeneous of degree one in K and L. We define $k = K/L$ and the per capita production $f(k)$ as

$$f(k) = \frac{F(K, L)}{L} = F(K/L, 1).$$

Note that $K = kL$ so,

$$K' = k'L + kL' = k'L + kgL.$$

This gives,

$$\begin{aligned}
&F(K, L) - C - \delta K = k'L + kgL, \\
\leftrightarrow &L(f(k) - c - \delta k) = L(k' + kg), \\
\rightarrow &k' = f(k) - c - \delta k - kg.
\end{aligned}$$

We assume that $u(c)$ is the per utility of per capital consumption $c = C/L$ then the optimization problem is,

$$\begin{aligned}
&\max \int_0^T u(c)dt, \\
&\text{s.t. } \dot{k} = f(k) - c - (\delta + g)k, \\
&\quad k(0) = k_0, k(T) = k_T.
\end{aligned}$$

The maximum principle gives,

$$\begin{aligned}
&u'(c) - \lambda = 0, \\
&\lambda' = (\rho + \delta - f'(k))\lambda, \\
&k' = f(k) - c - (\delta + g)k.
\end{aligned}$$

Draw a state-space diagram for this solution in the ck-plane.

Optimal investement

CONSIDER A PRICE taking firm that maximizes the discounted value of all future profits. The firm has to decide on the level of investment I but there are adjustment costs $\phi(I)$ such that $\phi'(I) \geq 0$ and $\phi(.)$ is convex. We assume that $\phi(0) = \phi'(0) = 0$. The capital stock $K(t)$ adjusted according to,

$$K'(t) = I - \delta K,$$

where $\delta \in (0,1)$ is the depreciation rate. The output for a level $K(t)$ of capital is given by $f(K(t))$. The firm solves the following problem.

$$\max \int_0^\infty e^{-rt}[f(K) - I - \phi(I)]dt,$$
$$\text{s.t. } K' = I - \delta K, K(0) \text{ given.}$$

The current value Hamiltonian is given by,

$$C = [f(K) - I - \phi(I)] + q[I - \delta K],$$

where q is the co-state variable.[96] The maximum principle gives,

$$\begin{aligned}
&-1 - \phi'(I) + q = 0,\\
&q' - rq = -f'(K) + \delta q,\\
&K' = I - \delta K,\\
&\lim_t e^{-rt} q(t) = 0.
\end{aligned}$$

Often the last transversality condition is replaced by $\lim e^{-rt}q(t)K(t) = 0$. Differentiating the first conditions with respect to time gives,

$$q' = \phi''(I)I'.$$

Substituting into the second condition,

$$\begin{aligned}
\phi''(I)I' &= -f'(K) + (\delta + r)(1 + \phi'(I)),\\
\rightarrow I' &= \frac{1}{\phi''(I)}[(\delta + r)(1 + \phi'(I)) - f'(K)].
\end{aligned}$$

Observe that if $\phi''(I) \rightarrow 0$ then I jumps immediately to the optimal investment value. However, due to the adjustment costs, investments are adjusted gradually to the optimal value. Draw a state-space diagram for this solution in the KI-plane.

The q-theory of investment defines the q value as the ratio of the value of an extra unit of capital over the replacement cost. If the ratio, called the Tobin-q is larger than 1, then it is optimal to increase the capital stock. In the steady state, $q' = 0$ so we have that,

$$q = \frac{f'(k)}{r + \delta} = 1 + \phi'(I^*) = 1 + \phi'(\delta K^*).$$

[96] The reason for the choice of q will become apparent soon.

This is close to one, except for the adjustment costs in the steady state, i.e. $\phi'(\delta K^*)$. The Tobin-q is often estimated using the ratio of the stock market price of a firm and the book value of the firm. If the value is larger than one it means that investments should increase as the value of installed capital is larger than the replacement cost. Notice however, that this measures the average q while in theory it is the marginal q that is relevant.

Exercises

Solve the following maximization problem using the maximum principle

- $V[y(t),u(t)] = \int_0^2 (y - u^2)dt$ with $y' = u, y(0) = 0$ and $y(2)$ free. (sol: $y = -t^2/4 + t$)
- $V[y(t),u(t)] = \int_0^2 (2y - 3u)dt$ with $y' = y + u$, $u \in [0,2]$, $y(0) = 4$ and $y(2)$ free.
- $V[y(t),u(t)] = \int_0^1 (-u^2)dt$ with $y' = y + u, y(0) = 1$ and $y(1) = 0$. (sol : $y = e^t/(1 - e^2) + e^{2-t}/(e^2 - 1)$)
- $V[y(t),u(t)] = \int_0^1 (-u^2)dt$ with $y' = y + u$, $y(0) = 1$ and $y(1) \geq 2$. (sol: $y = e^t$.)
- $V[y(t),u(t)] = \int_0^{20} (4y - u^2)e^{-0.25t}dt$ with $y' = -0.25y + u$, $y(0) = y_0$, $y(20)$ free. (sol: $y = (y_0 - 16 + 16/3e^{-10})e^{-0.25t} + 16 - (16/3)e^{0.5t-10}$)

Draw the phase diagram of the following problem (call λ the shadow price of the current value Hamiltonian).

- $\max \int_0^\infty (x - u^2)e^{-0.1t}dt$, $x' = -0.4x + u$, $x(0) = 1$, $x(\infty)$ free, $u \in (0,\infty)$. in the λx-space.
- $\max \int_0^\infty \frac{1}{1-\sigma} C^{1-\sigma} e^{-rt}dt$, $K' = aK - bK^2 - C$, $K(0) = K_0 > 0$ where $a > r > 0$ and $\sigma > 0$ and with $K(t) \geq 0$ for all t in the λK space.
- $\max \int_0^\infty (ax - 1/2u^2)e^{-rt}dt$, $x' = -bx + u$, $x(0) = x_0$, $x(\infty)$ free and all constants are positive in the λx-space.
- $\max \int_0^T \ln(C(t))e^{-rt}dt$, $K' = AK^\alpha - C$, $K(0) = K_0$, $K(T) = K_T$ where $r > 0$ and $A > 0$ and $\alpha \in (0,1)$ for all t in the CK-space.